Monotonicity in Complex Hyperbolic Triangle Groups

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1 Introduction

This purpose of these notes is to describe some experiments I recently did with complex hyperbolic triangle groups, and also to prove one easy result about them. More specifically, I am talking about the complex reflection triangle groups acting on the complex hyperbolic plane $\boldsymbol{CH}^2$. These groups are deformations of classic hyperbolic triangle groups acting on the real hyperbolic plane $\boldsymbol{H}^2$.

These notes are an expansion of some comments I made in my short paper [**S**] in the Proceedings of the 2002 ICM. At the time I wrote that paper, fewer people were interested in the kind of monotonicity phenenoma I will discuss here. Now that the topic is in the air, I did more extensive experiments and will formulate some crisp conjectures about them. Finally, I'll sketch the proof a partial result which gives additional credence to the conjectures.

In the rest of the introduction I will define enough terms so that the main conjectures are understandable. Following this introduction the rest of the notes are devoted to describing the experiments in enough detail that a competent programmer could repeat them.

The Complex Hyperbolic Plane: See [**G**] for an extensive treatment of complex hyperbolic geometry. The *complex hyperbolic plane* $\boldsymbol{CH}^2$ is the open unit ball in $\boldsymbol{C}^2$, equipped with a Riemannian metric (and in fact a

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Kahler metric) that is invariant under all complex projective transformations which perserve the unit ball. This metric is normalized to agree with the standard Euclidean metric at the origin. The group of such isometries is usually denoted $PU(2,1)$. The group $PU(2,1)$ has index 2 in the full isometry group of $\boldsymbol{CH}^2$; we get the full group by adjoining the map which is coordinate-wise complex conjugation. We are not going to care about these anti-holomorphic elements of $\mathrm{Isom}(\boldsymbol{CH}^2)$. The element $T \subset PU(2,1)$ is called *loxodromic* if there is some $d > 0$ such that T moves all points of $\boldsymbol{CH}^2$ at least d units, and *elliptic* if T fixes a point in $\boldsymbol{CH}^2$. Otherwise, T is called *parabolic*.

Complex Reflections: A *complex reflection* is an element of $PU(2,1)$ that is conjugate to the map $(z,w) \to (z,-w)$. The fixed point set of this particular complex reflection is the complex line $w=0$, and in general the fixed point set is some complex line. Conversely, given a complex line in $\boldsymbol{C}^2$ which intersects $\boldsymbol{CH}^2$, there is a unique complex reflection that fixes that line. Thus, complex reflections can be specified by complex lines.

Angles between Complex Lines: Suppose L_1 and L_2 are complex lines which intersect in a point of $\boldsymbol{CH}^2$. Then there is an isometry I such that $I(L_1)$ and $I(L_2)$ intersect at the origin, and both $I(L_1)$ and $I(L_2)$ intersect $\boldsymbol{R}^2$ in a line. The angle between L_1 and L_2 is defined to be the angle between the real lines $I(L_1)\cap\boldsymbol{R}^2$ and $I(L_2)\cap\boldsymbol{R}^2$. This definition can be given entirely in terms of $PU(2,1)$ invariant quantities. See the discussion around Equation 5.

Complex Hyperbolic Triangles: Call a triple $p_1,p_2,p_3 \in \boldsymbol{CH}^2$ *non-degenerate* if these points do not all lie in a complex line. In this case, these points define 3 distinct complex lines L_1,L_2,L_3. Here L_j is the complex line containing p_{j-1} and p_{j+1}, with indices taken mod 3. We let θ_j be the angle at p_j between L_{j-1},L_{j+1}. This configuration of points and lines is called a (n_1,n_2,n_3) triangle if $\theta_i = \pi/n_i$. Typically we use this notation only when n_1,n_2,n_3 are integers. Just as in the real hyperbolic case, a necessary and sufficient condition for the existence of such a configuration is that

$$\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}<1.$$

To avoid trivialities we take $n_1 \geq n_2 \geq n_3 \geq 3$. Thus, all choices are possible except $n_1=n_2=n_3=3$.

The space of triples of points is $12 = 4 \times 3$ dimensional. The group $PU(2,1)$ is 8 dimensional. Thus there is a 4-dimensional family of nondegenerate triples. The map $(p_1, p_2, p_3) \to (\theta_1, \theta_2, \theta_3)$ goes from the 4-dimensional moduli space into $\boldsymbol{R}^3$ and so the fibers are typically 1-dimensional. Thus, if we fix (n_1, n_2, n_3) we expect a 1-dimensional moduli space of each triple.

The Moduli Space: We fix an allowable triple (n_1, n_2, n_3) as above and keep with the notation already established. We have the associated triangle group generated by the reflections I_1, I_2, I_3 in the lines L_1, L_2, L_3. The product $I_{j-1} I_{j+1}$ has order n_j. Given a word w in the digits $\{1, 2, 3\}$, we let I_w denote the corresponding product of the generators. Thus, $I_{1213} = I_1 I_2 I_1 I_3$.

There is a canonical interval of parameters associated to (n_1, n_2, n_3). We always identify this interval with $[0, 1]$ and we use the variable $t \in [0, 1]$ to describe the corresponding groups. At the parameter $t = 0$ the three points p_1, p_2, p_3 lie $\boldsymbol{R}^2$. In this case the intersection $L_j \cap \boldsymbol{R}^2$ for $j = 1, 2, 3$ is just a real hyperbolic (n_1, n_2, n_3) triangle drawn in the Klein model. For all $t \in [0, 1)$ the elements I_w are all loxodromic for the following list of words:

- $w = abc$, where a, b, c are distinct digits.
- $w = abac$ where a, b, c are distinct digits.

This uniquely characterizes our intervl of paramaters. We call this interval the *preferred interval*.

We let $\Gamma(n_1, n_2, n_3; t)$ be the group corresponding to $t \in [0, 1]$. The parameter t is essentially an affine function of the distance between p_1 and p_2. I will say this more explicitly in the next chapter.

Discreteness Conjecture: In my 2002 ICM paper I conjectured that an (n_1, n_2, n_3) complex hyperbolic triangle group is discrete provided that it lies in the preferred interval. As we pass the the end of the interval, one of the elements just discussed instantly becomes elliptic, and so the preferred interval is the maximal interval of discreteness.

First Monotonicity Conjecture: Given $\gamma \in PU(2, 1)$, we define

$$\Delta(\gamma) = |z|^4 - 8\mathrm{Re}(z^3) + 18|z|^2 - 27,$$

where z is the trace of γ. The quantity $\Delta(\gamma)$ is called the *Goldman discriminant*. It is positive if and only if γ is loxodomic, and it serves as a proxy for the translation length of the isometry. Given an element $\gamma \in \Gamma(n_1, n_2, n_3; 0)$ we let γ_t denote the corresponding element of $\Gamma(n_1, n_2, n_3; t)$. In particular, $\gamma_0 = \gamma$. According to the Discreteness Conjecture, if $\Delta(\gamma_0) > 0$ then $\Delta(\gamma_t) > 0$ for all $t \in [0, 1)$.

Here is the first monotonicity conjecture: For any loxodromic element $\gamma \in \Gamma(n_1, n_2, n_3; 0)$, the quantity $f(t) = \Delta(\gamma_t)$ is monotone decreasing for $t \in [0, 1]$. This conjecture probably implies that the translation distance is also monotone decreasing, but I didn't explicitly check this.

Second Monotonicity Conjecture: The group $\Gamma(n_1, n_2, n_3; 0)$ is determined by the three initial points p_1, p_2, p_3. These points are the vertices of the triangle determining the group. Let V_0 denote the orbit of these points under $\Gamma(n_1, n_2, n_3; 0)$. The elements of V_0 are the fixed points of the elliptic elements in $\Gamma(...; 0)$. We define V_t in the same way. There is a canonical map from V_0 to V_t which sends the fixed point of a given elliptic of $\Gamma(...; 0)$ to the fixed point of the corresponding elliptic in $\Gamma(...; t)$.

A better way to say this is as follows: For each pair $v_0, w_0 \in V_0$ there is a 1-parameter family v_t, w_t. We let $f(v, w, t)$ denote the distance between v_t and w_t. Our second conjecture is that the function $f(v, w, t)$ is non-increasing for all choices of n_1, n_2, n_3 and v, w.

Applications: The Monotonicity Conjectures would go a long way towards proving the Discreteness Conjecture. Essentially they reduce the Discreteness Conjecture to the statement that $\Gamma(n_1, n_2, n_3; 1)$ is discrete and faithful. Here are some specially nice examples. Let $n_1, n_2, n_3 \in \{3, 4, 6\}$. Then $\Gamma(n_1, n_2, n_3; 1)$ has the property that the trace of every word has a half-integer real part. (For some choices you get integers rather than half-integers.) In all these cases, the corresponding groups are discrete. So, if we knew either Monotonicity Conjecture, we could conclude the Discreteness Conjecture for these 9 cases. I suspect that a similar easy argument works for any of the arithmetic triangle groups.

Another application is that the Hausdorff dimension of the limit set $\Lambda(n_1, n_2, n_3; t)$ is monotone increasing in t. This follows from the connection between the Hausdorff dimension of the limit set and the Poincare series.

Partial Results: Call a triple (n_1, n_2, n_3) *partially even* if at least one of the integers is even. In §3 I will sketch the proof that the First Monotonicity Conjecture holds for infinitely many words (which are not powers of each other) with respect to a partially even triple. I'll prove a similar kind of statement about the Second Monotonicity Conjecture. It turns out that, in the partially even case, many of the fixed points of finite order elements are organized into totally regular infinite trees, each of which is contained in a totally real, totally geodesic slice of $\boldsymbol{CH}^2$. (When all the integers are even, all the fixed points are organized into these trees.) Essentially, both conjectures are true if we just restrict attention to any of these trees.

These modest results give some theoretical credence to the conjectures, and also reveal some of the geometric structure of these groups. I am not sure how to promote these partial results to a proof in general. However, the geometry is really beautiful and merits further study.

2 Computations

2.1 Defining Triangle Groups

The basic object for us is the Hermitian form on $\boldsymbol{C}^{2,1}$ given by

$$\langle V, W\rangle = v_1\overline{w}_1 + v_2\overline{w}_2 - v_3\overline{w}_3. \tag{1}$$

Here $V = (v_1, v_2, v_3)$ and likewise for W.

We let $\boldsymbol{C}^{2,1}_-$ denote the subset of vectors V such that $\langle V, V\rangle < 0$. These are called the *negative vectors*. The *positive vectors* are defined similarly.

There is a map Π from $\boldsymbol{C}^{2,1}$ to $\boldsymbol{CP}^2$, namely

$$\Pi(V) = \left(\frac{v_1}{v_3}, \frac{v_2}{v_3}\right). \tag{2}$$

$\boldsymbol{CH}^2$, the open unit ball in $\boldsymbol{C}^2$ is the image of $\boldsymbol{C}^{2,1}_-$ under this map.

Each positive vector W defines a complex line L_W in $\boldsymbol{CH}^2$, namely $\Pi(W^\perp)$ where $W^\perp$ consists of vectors V such that $\langle V, W\rangle = 0$. The complex reflection I_W fixing L_W has the formula

$$I_W(V) = V - 2\frac{\langle V, W\rangle}{\langle W, W\rangle}W. \tag{3}$$

Given U, V define the *box product*

$$U \boxtimes V = (\overline{u_3v_2 - u_2v_3},\ \overline{u_1v_3 - u_3v_1},\ \overline{u_1v_2 - u_2v_1}). \tag{4}$$

This vector is such that $\langle U, U \boxtimes V\rangle = \langle V, U \boxtimes V\rangle = 0$.

Here is one more useful function. When V, W are positive vectors the angle θ between the complex lines they define satisfies the identity $\cos^2(\theta) = D(V, W)$, where

$$D(V, W) = \frac{\langle V, W\rangle\langle W, V\rangle}{\langle V, V\rangle\langle W, W\rangle}. \tag{5}$$

When V, W are negative vectors the distance d between satisfies $\cosh^2(d) = D(V, W)$.

Starting with the points p_1, p_2, p_3 (represented by negative vectors) we define positive vectors $W_j = p_{j-1} \boxtimes p_{j+1}$. We then define the complex reflections I_1, I_2, I_3 so that $I_j = I_{W_j}$. These respectively fix the complex lines L_1, L_2, L_3, where $L_j = \Pi(W_j^\perp)$.

2.2 Finding the Triangle Groups

Given (n_1, n_2, n_3) we let $\theta_1, \theta_2, \theta_3$ be the angles, namely $\theta_j = \pi/n_j$. In this section we explain how we compute (approximately) the preferred interval with respect to (n_1, n_2, n_3).

We first define

$$a_{\max} = \sqrt{1 - \alpha^{-2}}, \qquad \alpha = \frac{\cos(\theta_1)\cos(\theta_3) + \cos(\theta_2)}{\sin(\theta_1)\sin(\theta_3)}. \tag{6}$$

Next we define points

$$p_1 = (0, 0), \qquad p_2 = (a, 0), \qquad p_3 = b(\cos(\theta_1), \sin(\theta_1)). \tag{7}$$

When $a = a_{\max}$ and $b \in \boldsymbol{C}$ is suitably chosen, we have the totally real triangle with angles $\theta_1, \theta_2, \theta_2$. The remaining triangles with these angles have $a \in (0, a_{\max})$. When a is sufficiently close to 0 the triangle does not exist.

Now we fix $a \in (0, a_{\max})$. By construction, the angle at p_1 is θ_1. For $x \in [0, 1]$ we define

$$b(x) = \frac{1}{C + \rho e^{\pi i x}}, \qquad C = \frac{\cos(\theta_1)}{a} \qquad \rho = \frac{(\sqrt{1 - a^2})\sin(\theta_1)}{a\tan(\theta_2)}. \tag{8}$$

That is, we take a suitable chosen number on the circle of radius ρ centered at C and we invert this number to get $b(x)$. Any choice of x gives us a triple $(p_1, p_2, p_3(x))$ with the angle at p_2 equal to θ_2. Since we want to have a point in $\boldsymbol{CH}^2$ we only do this when $|b(x)| < 1$.

Bisection Algorithm: We seek a value $x = x(a)$ such that the angle at $p_3(x)$ is θ_3. We define

$$f_a(x) = \theta - \theta_3(x), \tag{9}$$

Where $\theta_3(x)$ is the angle at $p_3(x)$.

If $f(0)$ and $f(1)$ have the same sign, we declare the algorithm a failure. Otherwise we start with $(x_1, x_2, x_3) = (1, 0, 1/2)$. By assumption $f(x_1)$ and $f(x_2)$ have opposite signs. At this poin we check the sign of $f(x_3)$. Then...

- If $f(x_1)$ and $f(x_3)$ have opposite signs we set $y_1 = x_1$ and $y_2 = x_3$ and $y_3 = (y_1 + y_2)/2$. Then we redefine $x_j = y_j$ for $j = 1, 2, 3$.

- If $f(x_2)$ and $f(x_3)$ have opposite signs we set $y_1 = x_3$ and $y_2 = x_2$ and $y_3 = (y_1 + y_2)/2$. Then we redefine $x_j = y_j$ for $j = 1, 2, 3$.

Now we have an interval half the size on which $f(x_1)$ and $f(x_2)$ have opposite signs. We iterate this 100 times and this gives us a value x_3 for which $f(x_3)$ is very near 0. We then set $p_3 = p_3(x_3)$ and stop.

The Range of Values: When $a < a_{\max}$ is sufficiently close to $a_{\max}$, the Bisection Algorithm works, and we get our triangle group. When a is sufficiently close to 0 the Bisection Algorithm fails. We run a second bisection algorithm to find the value $a_{\min}$ such that the interval $[a_{\min}, a_{\max}]$ precisely gives the preferred interval. Here is how this works. We define $(a_1, a_2, a_3) = (0, 1, 1/2)$ as above.

Our running assumption is that the group $\Gamma(a_2)$ exists and is in the preferred interval. This is true when $a_2 = 1$. The group $\Gamma(0)$ does not exist. We check whether $\Gamma(a_3)$ exists and is in the preferred interval...

- If so, then we set $b_1 = a_1$ and $b_2 = a_3$ and $b_3 = (b_1 + b_2)/2$.
- If not, then we set $b_1 = a_3$ and $b_2 = a_2$ and $b_3 = (b_1 + b_2)/2$.

Then we redefine $a_j = b_j$ for $j = 1, 2, 3$. We repeat this 100 times and this gives us a very good approximation to $a_{\min}$.

The Final Parametrization: Given $t \in [0, 1]$ we let $a = (1-a)a_{\max} + ta_{\min}$. We then run the Bisection Algorithm to find $b = b(a)$ and this gives us the group we are call $\Gamma(n_1, n_2, n_3; t)$.

2.3 The Experiments

Here are the basic definitions we need:

- Let $G(M)$ denote the list of all allowable triples in lexicographic order. Thus, for instance, $G(4) = \{433, 443, 444\}$. The set $G(12)$ has 219 triples.
- Let $W(N)$ denote the set of all words of length at most N. For instance $W(3) = \{123, 132, 213, 231, 312, 321\}$. The set $W(14)$ has 49140 words.
- For any K let I_k denote the k intervals of length $1/K$ in $[0, 1]$. For instance $I_2 = \{[0, 1/2], [1/2, 1]\}$.

Let $\Delta(n_1, n_2, n_3; t; w)$ denote the Goldman discriminant of I_w with respect to the group $\Gamma(n_1, n_2, n_2; t)$. When the triple (n_1, n_2, n_3) is fixed we shorten this notation to $\Delta(t; w)$.

Given M, N, K we consider all choices $(n_1, n_2, n_3) \in G(N)$ and $w \in W(N)$ and $[t_0, t_1] \in I_K$. For each choice we do the following:

- We first check that w is loxodromic. This means that the word I_w is loxodromic in $\Gamma(n_1, n_2, n_3; 0)$, the real group. If not, we ignore w.

- We check numerically that $\Delta(t; w)$ is not constant. Our check has a tolerance of about 10^{-4}, which takes care of the round-off error our method produces for big groups and long words. If $\Delta(t; w)$ is constant, we ignore w.

- If we have not ignored w we check that $\Delta(t_0; w) > \Delta(t_1, w)$.

We say that the triple (M, N, K) is *discriminant good* if this check works for all relevant choices.

Now we turn to a discussion of the vertices. We keep the same notation as above. For each word w and each triple $(i, j) \in \{1, 2, 3\}^2$ we consider the pair of vertices

$$v = p_i, \qquad w = I_w(p_j). \tag{10}$$

The vertices v, w are functions of the quintuple $(n_1, n_2, n_3; t; v, w)$. We define $D(n_1, n_2, n_3; t; v, w)$ to be the quantity $D(V, W)$ given in Equation 5. Here V and W are negative vectors representing v and w respectively. Again, $D(V, W)$ is a monotone increasing function of the distance between v and w.

For exactly the same words, groups, and intervals, and for every pair (i, j) we check that $D(t_0; v, w) > D(t_1; v, w)$ for all relevant cases. If the check always works, we call the group (M, N, K) *vertex good*.

Using several hours of computation on my 2016 IMAC, I checked, modulo roundoff error, that the triple $(12, 14, 32)$ is both discriminant good and vertex good. This check takes several hours to run.

3 A Partial Result

3.1 Some Structure

There is a graph associated to the group $\Gamma(n_1, n_2, n_3; t)$. The vertices are the fixes point sets of the elements conjugate to $I_j I_k$. The edges are geodesic segments that connect up adjacent fixed points. This graph is naturally isometric to the 1-skeleton of the tiling of the real hyperbolic plane associated to $\Gamma(n_1, n_2, n_3; 0)$. Call this graph Λ. We call a vertex in Λ *even* if its degree is twice an even number. Note that n_j is even if and only if v_j is even. When Λ is associated to a partially even group, there are infinitely many even vertices.

Even Flowers: The *flower* of a vertex of Λ is the union of v and all the adjacent vertices. Call this flower $F(v)$. There are $2n_j$ vertices in $F(v)$ aside from v. These vertices may be split into 2 groups, depending on their parity in their cyclic order. Thus, if $v, w_1, ..., w_{2n}$ are the vertices in the flower, then we have the two even flowers $F_1(v) = v, w_1, w_3, ...$ and $F_2(v) = v, w_2, w_4, ...$ Each of these flowers has $n + 1$ vertices, and v is central to both. The remaining vertices we call *peripheral*. We call $F_1(v)$ and $F_2(v)$ *even flowers*.

Here is the first basic fact: The vertices of a even flower all lie in the same $\boldsymbol{R}$-slice. That is, there is a totally real geodesic plane containing all the vertices of a even flower. In general, the two $\boldsymbol{R}$-slices so defined at each vertex are distinct from each other. This just amounts to a calculation.

Hooked Even Flowers: We concentrate on even flowers now, and we change our notation so that $(v, w_1, w_2, ...)$ denotes an even flower. We call two even flowers

$$(v, w_1, ..., w_k), \qquad (v', w'_1, ..., w'_k)$$

hooked if they have exactly one vertex in common, say $w_1 = w'_1$, and the following two conditions arise:

- The triple
$$v, w_1 = w'_1, v'$$
lies on a geodesic segment.

- $w_1 = w'_1$ is an even vertex.

In any partially even group, there are many hooked even flowers. Here is how they arise. Let's say that n_1 is even. Then the element

$$J = (I_{n_2} I_{n_3})^{n_1/2}$$

is an involution. We take any even flower F that has v_1 as a peripheral vertex and then we observe that F and $J(F)$ are hooked. The common peripheral vertex in this case is v_1. This picture occurs in the 2-neighborhood of any even vertex.

Here is the second basic fact: Two hooked even flowers lie in the same $\boldsymbol{R}$-slice. This is another calculation, but I think that probably it just follows from symmetry and special properties of the involution J mentioned above.

Even Trees: We call two even flowers F_0, F_n *equivalent* if there is a sequence $F_0, F_1, ..., F_n$ such that F_j and F_{j+1} are hooked for all j. Two equivalent flowers lie in the same $\boldsymbol{R}$-slice. We define the *grand continuation* of the flower to be the union of all the edges and vertices of all the flowers in its equivalence class. The grand continuation of a flower is a tree whose interior vertices all have the same degree and whose interior edges all have the same length. The leaves of the tree are odd vertices in Λ. The interior edges have even vertices for midpoints.

The nicest case to consider is the situation when all of (n_1, n_2, n_3) are even. In this case, all the grand continuations of even flowers are infinite regular trees. Even in the case when just one of the integers is even, there are infinitely many grand continuations which are infinite regular trees.

3.2 Monotonicity of Real Regular Trees

As a prelude to proving our result, we consider the situation for infinite regular trees embedded in the real hyperbolic plane. Up to isometry, such a tree is determined by a pair of integers (d, L). Here d is the degree and L is the length of the edges.

The basic fact is that these regular trees are completely monotone. If we consider the two trees $T(d, L)$ and $T(d, L')$ with $L' > L$ then the distance between every pair of vertices v', w' in the latter tree is greater than the distance between the corresponding vertices v, w in the former. More is true: There is a distance non-increasing map from $\boldsymbol{H}^2$ to $\boldsymbol{H}^2$ which maps $T(d', L')$ to $T(d, L)$. I think that this is a well-known result.

I remember hearing a proof this result from my friend Peter Doyle, and also I think that these kinds of considerations came up in the work of Goldman, Minsky, (and maybe Margulis) on Lorentzian space times. The point is that they have to do with *totally positive deformations*. I can't remember this too well and I don't think that these guys ever published the paper. Also, it probably arises in the proof by Danciger, Gueritaud, and Kassel of the crooked planes conjecture.

Anyway, here is a sketch of the proof. Let us work in the copy of $\boldsymbol{H}^2$ which contains $T(d, L)$. We define a family of perpendicular bisectors to the edges of the tree. These turn out to be disjoint from each other. Now we go a kind of grafting operation where we replace each of these bisectors with an infinite geodesic strip which, at its narrowest, has width $L' - L$. The boundaries of the strip should be perpendicular to the relevant edge of the tree. When we are done, we have an exact copy of $\boldsymbol{H}^2$ (!) The new copy has $T(d, L')$ embedded in it rather than $T(d, L)$. The distance non-increasing map just collapses the strips.

3.3 Putting it Together

Now suppose we have a partially even family $\Gamma(n_1, n_2, n_3, t)$. We we increase t our infinite regular trees keep their degree but their edges get shorter. Thus, the distance between any pair of vertices within such a tree decreases monotonically. This is the infinite list of pairs for which the Second Monotonicity Conjecture holds.

It is worth mentioning that some cases of this are especially beautiful. For instance, in the $(4, 4, 4)$ case, in the last (conjecturally) discrete group in the family, these infinite regular trees develop cusps: They are just barely embedded: Their accumulation set on the ideal boundary is an entire $\boldsymbol{R}$-circle. These $\boldsymbol{R}$-circles lie in the limit set of the group, and so the limit set in this case has a dense family of $\boldsymbol{R}$-circles. This is a kind of $\boldsymbol{R}$-circle analogue of a circle packing limit set such as the Apollonian gasket.

For the First Monotonicity Conjecture, consider any word which preserves one of our infinite trees. Such a word also preserves the $\boldsymbol{R}$-slice containing it. Going back to the analysis of the real case, there is a distance non-increasing map of the $\boldsymbol{R}$-slice to itself which conjugates the word associated to $\Gamma(n_1, n_2, n_2; t)$ to the corresponding word in $\Gamma(n_1, n_2, n_3, t')$ with $t' > t$. I didn't do the calculation, but the monotonicity of the Goldman discriiminant should be pretty obvious from this.

4 References

[**G**] W. M. Goldman, *Complex Hyperbolic Geometry*, Oxford University Press (1999)

[**S**] R. E. Schwartz, *Complex Hyperbolic Triangle Groups*, Proceedings of the International Congress of Mathematicians (2002)