Kähler Differentials from a Geometric Point of View

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Kähler differentials in algebraic geometry play the role of differential 1-forms in geometry. In this note we define vector fields and 1-forms from the perspective of commutative algebra, and then draw connections between the algebraic definitions and the geometric counterparts.

1 Derivations and Vector Fields

A derivation from a ring to a module generalizes the notion of a differential operator, in that it is a k-linear map that satisfies the product rule from calculus.

Definition 1.1. Let k be a field, S a k-algebra, and M an S-module. A k-linear map $D : S \to M$ is called a *derivation* if it satisfies the *Leibniz Rule*

$$D(fg) = fD(g) + gD(f)$$

for all $f, g \in S$.

The collection $\mathrm{Der}_k(S, M)$ of all k-linear derivations $S \to M$ has the structure of an S-module, given by

$$(D_1 + D_2)(f) := D_1(f) + D_2(f),$$
$$(fD)(g) := f \cdot D(g)$$

for $D_1, D_2 \in \mathrm{Der}_k(S, M)$ and $f, g \in S$.

Remark 1.1. Derivations D always satisfy $D(a) = 0$ for all $a \in k$. To see this, note that $D(1) = D(1 \cdot 1) = 1D(1)+1D(1)$ by the Leibniz rule, so that $D(1) = 0$. It then follows from k-linearity that $D(a) = aD(1) = 0$. Geometrically this says that "the derivatives of constant functions are zero."

Example 1.1.1. When $S = k[x, y]$, the x-partial derivative operator $\partial_1 : k[x, y] \to k[x, y]$ is a k-linear derivation from $k[x, y]$ to itself. For example, $\partial_1(xy) = y$ and $\partial_1(x^2y^2) = 2xy^2$.

Example 1.1.2. Let N be a smooth manifold and let $S' = C^\infty(N)$ be the ring of smooth, real-valued functions on N. An element $X \in \mathrm{Der}_{\mathbb{R}}(C^\infty(N), C^\infty(N))$ is called a *smooth vector field* on N. At each point $p \in N$, the vector field X determines a functional $X_p : C^\infty(N) \to \mathbb{R}$ by the rule $X_p(f) := X(f)(p)$. The functional X_p is called a *tangent vector* based at p, since X_p is an $\mathbb{R}$-linear map satisfying the Leibniz rule

$$X_p(fg) = f(p)X_p(g) + g(p)X_p(f)$$

for all $f, g \in C^\infty(N)$.

One thinks of the quantity $X_p(f)$ as being the directional derivative of f at p in the direction of X_p. The $\mathbb{R}$-vector space of all tangent vectors is the *tangent space* T_pN to N at p. A vector field X can then be thought of as a choice of tangent vector in T_pN that "varies smoothly" in p.

The first example is a special case of the second, since the partial derivative operator $\partial/\partial x$ can be identified with a unit tangent vector field pointing in the x-direction at each point of $\mathrm{Spec}(k[x, y])$.

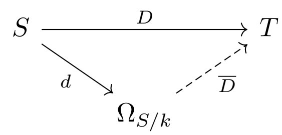
Example 1.1.3. Given a smooth function $f : N \to \mathbb{R}$, the *differential* of f, denoted df, is a certain "smoothly-varying" collection of linear functionals $df_p : T_pN \to \mathbb{R}$. Explicitly, df sends the tangent vector $V_p \in T_pN$ to $df_p(V_p)$, the directional derivative of f at p in the direction V_p. The differential df is an example of a *covector field*, in that it is in some sense dual to a vector field (we will see this later).

The *exterior derivative* operator $d : C^\infty(N) \to \{\text{Covector Fields}\}$ that sends the smooth function f to its differential df is a derivation: the map d is $\mathbb{R}$-linear and satisfies the Leibniz rule $d(fg) = fdg + gdf$.

2 Differentials and 1-Forms

We algebraically describe differentials of smooth functions and the exterior derivative operator by constructing a *universal* S-module $\Omega_{S/k}$ and derivation $d : S \to \Omega_{S/k}$. The module $\Omega_{S/k}$ is generated by expressions of the form $\{df : f \in S\}$, with relations imposed by the Leibniz rule. Thinking of the elements of S as "functions" on $\mathrm{Spec}(S)$, the module of differentials $\Omega_{S/k}$ consists of the "1-forms" on $\mathrm{Spec}(S)$.

Definition 2.1. Let S be a k-algebra. The *module of differentials* of S over k is the S-module $\Omega_{S/k}$ and k-linear derivation $d : S \to \Omega_{S/k}$ satisfying the following universal property: If T is an S-module and $D : S \to T$ is a k-linear derivation, then there exists a *unique* S-linear map $\overline{D} : \Omega_{S/k} \to T$ such that the diagram



commutes.

One explicit construction of $\Omega_{S/k}$ is as a quotient of the free S-module $\bigoplus_{f \in S} S[df]$ by the relations $[d(af + bg)] = a[df] + b[dg]$ and $[d(fg)] = f[dg] + g[df]$ for all $f, g \in S$ and for all $a, b \in k$. The quotient map $d : S \to \bigoplus_{f \in S} Sdf \to \Omega_{S/k}$ sending f to its "differential" df is then a derivation by construction. The induced S-linear map $\overline{D} : \Omega_{S/k} \to T$ from the universal property is given by $df \mapsto Df$ for all $f \in S$.

Example 2.1.1. If $S = k[x_1, \ldots, x_r]$ is a polynomial ring in r variables over k, then $\Omega_{S/k} \cong \bigoplus_{i=1}^{r} Sdx_i$.

Proof. Since S is generated as a k-algebra by $\{x_1, \ldots, x_r\}$, repeated use of the Leibniz rule shows that $\Omega_{S/k}$ is generated as an S module by $\{dx_1, \ldots, dx_r\}$. Hence there is a surjection $S^r \to \Omega_{S/k}$ that sends the ith basis vector e_i to dx_i.

On the other hand, the partial derivative $\partial_i : S \to S$ with respect to the variable x_i is a k-linear derivation, so by the universal property there is an induced map $\overline{\partial_i} : \Omega_{S/k} \to S$ sending dx_i to 1 and all other dx_j to 0. Packaging the first partials together gives an S-module map $(\partial_i)_{i=1}^{r} : \Omega_{S/k} \to S^r$ that is an inverse map. □

As a corollary, in $\Omega_{S/k}$ the differential df can be expressed as $df = \sum \frac{\partial f}{\partial x_i} dx_i$. This coincides with the usual definition of differential when $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function on Euclidean n-space.

A rephrasing of the universal property is that there is a natural isomorphism

$$\mathrm{Der}_k(S, M) \cong \mathrm{Hom}_S(\Omega_{S/k}, M)$$

between the endofunctors $\mathrm{Der}_k(S, -)$ and $\mathrm{Hom}_S(\Omega_{S/k}, -)$ on the category of S-modules.

Example 2.1.2. Again let N be a smooth manifold and $S' = C^\infty(N)$ the ring of smooth functions on N. Plugging in S' for M yields that *the S'-linear dual of $\Omega_{S'/k}$ is* $\mathrm{Der}_k(S', S')$. This is one sense in which differentials are the dual objects to the vector fields defined previously.

Given a vector field $X \in \mathrm{Der}_k(S', S')$, the universal property of $\Omega_{S'/k}$ induces a functional $\lambda_X : \Omega_{S'/k} \to S'$ sending df to the function $X(f)$. Conversely, a functional $\lambda : \Omega_{S'/k} \to S'$ determines a smooth vector field $X_\lambda : S' \to S'$ by the formula $X_\lambda(f) := \lambda(df)$.

Remark 2.1. The construction of the module of differentials $\Omega_{S/k}$ is functorial in the following sense: let S and S' be k-algebras, let $d : S \to \Omega_{S/k}$ and $d' : S' \to \Omega_{S'/k}$ be the two universal derivations, and let $\phi : S \to S'$ be a map that fixes k. The composition $d' \circ \phi$ is a k-derivation from S to $\Omega_{S'/k}$, so by the universal property there is an induced S-module map $\phi : \Omega_{S/k} \to \Omega_{S'/k}$ sending fdg to $\phi(f)d\phi(g)$. This is a restatement of the fact that exterior derivative commutes with pullbacks.

References

[1] David Eisenbud. *Commutative Algebra with a View Towards Algebraic Geometry.* Spinger-Verlang, New York, 1995.

[2] *The CRing Project*, 2010. `http://math.uchicago.edu/~amathew/CRing.pdf`