Preface

The title of the book sounds a bit mysterious. Why should anyone read this book if it presents the subject in a wrong way? What is particularly done "wrong" in the book?

Before answering these questions, let me first describe the target audience of this text. This book appeared as lecture notes for the course "Honors Linear Algebra". It supposed to be a *first* linear algebra course for mathematically advanced students. It is intended for a student who, while not yet very familiar with abstract reasoning, is willing to study more rigorous mathematics than what is presented in a "cookbook style" calculus type course. Besides being a first course in linear algebra it is also supposed to be a first course introducing a student to *rigorous* proof, formal definitions—in short, to the style of modern theoretical (abstract) mathematics. The target audience explains the very specific blend of elementary ideas and concrete examples, which are usually presented in introductory linear algebra texts with more abstract definitions and constructions typical for advanced books.

Another specific of the book is that it is not written by or for an algebraist. So, I tried to emphasize the topics that are important for analysis, geometry, probability, etc., and did not include some traditional topics. For example, I am only considering vector spaces over the fields of real or complex numbers. Linear spaces over other fields are not considered at all, since I feel time required to introduce and explain abstract fields would be better spent on some more classical topics, which will be required in other disciplines. And later, when the students study general fields in an abstract algebra course they will understand that many of the constructions studied in this book will also work for general fields.

Also, I treat only finite-dimensional spaces in this book and a basis always means a finite basis. The reason is that it is impossible to say something non-trivial about infinite-dimensional spaces without introducing convergence, norms, completeness etc., i.e. the basics of functional analysis. And this is definitely a subject for a separate course (text). So, I do not consider infinite Hamel bases here: they are not needed in most applications to analysis and geometry, and I feel they belong in an abstract algebra course.

Notes for the instructor. There are several details that distinguish this text from standard advanced linear algebra textbooks. First concerns the definitions of bases, linearly independent, and generating sets. In the book I first define a basis as a system with the property that any vector admits a unique representation as a linear combination. And then linear independence and generating system properties appear naturally as halves of the basis property, one being uniqueness and the other being existence of the representation.

The reason for this approach is that I feel the concept of a basis is a much more important notion than linear independence: in most applications we really do not care about linear independence, we need a system to be a basis. For example, when solving a homogeneous system, we are not just looking for linearly independent solutions, but for the correct number of linearly independent solutions, i.e. for a basis in the solution space.

And it is easy to explain to students, why bases are important: they allow us to introduce coordinates, and work with $\mathbb{R}^n$ (or $\mathbb{C}^n$) instead of working with an abstract vector space. Furthermore, we need coordinates to perform computations using computers, and computers are well adapted to working with matrices. Also, I really do not know a simple motivation for the notion of linear independence.

Another detail is that I introduce linear transformations before teaching how to solve linear systems. A disadvantage is that we did not prove until Chapter 2 that only a square matrix can be invertible as well as some other important facts. However, having already defined linear transformation allows more systematic presentation of row reduction. Also, I spend a lot of time (two sections) motivating matrix multiplication. I hope that I explained well why such a strange looking rule of multiplication is, in fact, a very natural one, and we really do not have any choice here.

Many important facts about bases, linear transformations, etc., like the fact that any two bases in a vector space have the same number of vectors, are proved in Chapter 2 by counting pivots in the row reduction. While most of these facts have "coordinate free" proofs, formally not involving Gaussian elimination, a careful analysis of the proofs reveals that the Gaussian elimination and counting of the pivots do not disappear, they are just hidden in most of the proofs. So, instead of presenting very elegant (but not easy for a beginner to understand) “coordinate-free” proofs, which are typically presented in advanced linear algebra books, we use “row reduction” proofs, more common for the “calculus type” texts. The advantage here is that it is easy to see the common idea behind all the proofs, and such proofs are easier to understand and to remember for a reader who is not very mathematically sophisticated.

I also present in Section 8 of Chapter 2 a simple and easy to remember formalism for the change of basis formula.

Chapter 3 deals with determinants. I spent a lot of time presenting a motivation for the determinant, and only much later give formal definitions. Determinants are introduced as a way to compute volumes. It is shown that if we allow signed volumes, to make the determinant linear in each column (and at that point students should be well aware that the linearity helps a lot, and that allowing negative volumes is a very small price to pay for it), and assume some very natural properties, then we do not have any choice and arrive to the classical definition of the determinant. I would like to emphasize that initially I do not postulate antisymmetry of the determinant; I deduce it from other very natural properties of volume.

Note, that while formally in Chapters 1–3 I was dealing mainly with real spaces, everything there holds for complex spaces, and moreover, even for the spaces over arbitrary fields.

Chapter 4 is an introduction to spectral theory, and that is where the complex space $\mathbb{C}^n$ naturally appears. It was formally defined in the beginning of the book, and the definition of a complex vector space was also given there, but before Chapter 4 the main object was the real space $\mathbb{R}^n$. Now the appearance of complex eigenvalues shows that for spectral theory the most natural space is the complex space $\mathbb{C}^n$, even if we are initially dealing with real matrices (operators in real spaces). The main accent here is on the diagonalization, and the notion of a basis of eigesnspaces is also introduced.

Chapter 5 dealing with inner product spaces comes after spectral theory, because I wanted to do both the complex and the real cases simultaneously, and spectral theory provides a strong motivation for complex spaces. Other then the motivation, Chapters 4 and 5 do not depend on each other, and an instructor may do Chapter 5 first.

Although I present the Jordan canonical form in Chapter 9, I usually do not have time to cover it during a one-semester course. I prefer to spend more time on topics discussed in Chapters 6 and 7 such as diagonalization of normal and self-adjoint operators, polar and singular values decomposition, the structure of orthogonal matrices and orientation, and the theory of quadratic forms.

I feel that these topics are more important for applications, then the Jordan canonical form, despite the definite beauty of the latter. However, I added Chapter 9 so the instructor may skip some of the topics in Chapters 6 and 7 and present the Jordan Decomposition Theorem instead.

I also included (new for 2009) Chapter 8, dealing with dual spaces and tensors. I feel that the material there, especially sections about tensors, is a bit too advanced for a first year linear algebra course, but some topics (for example, change of coordinates in the dual space) can be easily included in the syllabus. And it can be used as an introduction to tensors in a more advanced course. Note, that the results presented in this chapter are true for an arbitrary field.

I had tried to present the material in the book rather informally, preferring intuitive geometric reasoning to formal algebraic manipulations, so to a purist the book may seem not sufficiently rigorous. Throughout the book I usually (when it does not lead to the confusion) identify a linear transformation and its matrix. This allows for a simpler notation, and I feel that overemphasizing the difference between a transformation and its matrix may confuse an inexperienced student. Only when the difference is crucial, for example when analyzing how the matrix of a transformation changes under the change of the basis, I use a special notation to distinguish between a transformation and its matrix.

Contents