on a train w/ no wifi

$\Rightarrow$ no fancy videos today :(

§14.7 Extreme Values and Saddle Points

$f(x,y)$ function of two variables

want to find maximum and minimum values of f

(together, called extreme values or extrema of f)

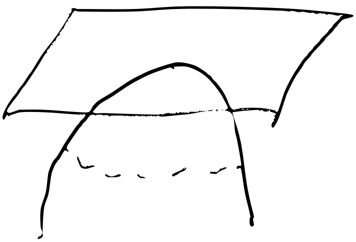
e.g. least cost, most power, ...

Def Let $f(x,y)$ be defined on a region containing (a,b)

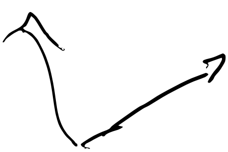
(a,b) is a local maximum of f if $f(a,b) \geq f(x,y)$ for all (x,y) in some open disk centered @ (a,b)

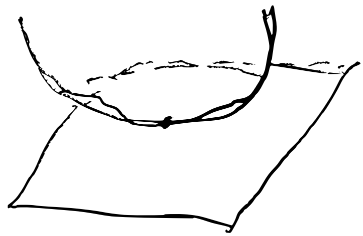
(a,b) is a local minimum of f if $f(a,b) \leq f(x,y)$ for all (x,y) in some open disk centered @ (a,b).

Graphically: if $z=f(x,y)$,



max = "peak"





min = "valley"

in both cases: looks like tangent plane is <u>horizontal</u>

why?

<u>Thm</u> (First derivative test)

If f has a local max/min @ (a,b), <u>and</u> $f_x(a,b)$ and $f_y(a,b)$ exist, then

$f_x(a,b)=0$ and $f_y(a,b)=0$.

Proof Bootstrap from 1d case!

If $f(x,y)$ has a local max (min) @ (a,b), then

$$g(x) = f(x,b)$$

has a local max (min) @ a.

$$\Rightarrow g'(a) = 0$$ (by single-variable calculus)

but $g'(a) = f_x(a,b)$, so $f_x(a,b) = 0$.

Same argument shows

$$f_y(a,b) = 0.$$

Def An interior point (a,b) of the domain of f is a critical point of f if

$[$both $f_x(a,b)=0$ and $f_y(a,b)=0]$

OR $[$at least one of $f_x(a,b)$ or $f_y(a,b)$ is undefined$]$

So to find local extrema, first look for critical points.

ex Find the critical points of $f(x,y) = x^2 + 2xy$

$f_x = 2x + 2y$] need $x=-y$ for this to be 0

$f_y = 2x$] need $x=0$ for this to be 0

so only critical point is $(0,0)$.

but this is not a local min or max!

- if we fix $y=0$, then $f(x,0) = x^2 \geq 0 = f(0,0)$
  so could only be a local min
- but for x small and $y<0$, $f(x,y)<0$
  $\Rightarrow$ not a local min

Def A critical point (a,b) of f is a saddle point if every open disk centered @ (a,b) contains points with $f(x,y) > f(a,b)$ and also points (x,y) with $f(x,y) < f(a,b)$.

(This is the higher-dimensional analogue of inflection points from single-variable calculus.)

e.g. $(0,0)$ is a saddle point of $f(x,y) = x^2 + 2xy$.

In this case, we could analyze the critical point $(0,0)$ by direct inspection. But what do we do in general?

<u>Thm</u> (Second derivative test)

Suppose $f(x,y)$ and its first (f_x, f_y) and second (f_{xx}, f_{xy}, f_{yy}) partial derivatives are cts throughout a disk centered @ (a,b), and

$$f_x(a,b) = 0 = f_y(a,b).$$

Then:

(a) f has a <u>local max</u> @ (a,b)

if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$

at (a,b)

(b) f has a <u>local min</u> @ (a,b) if

$f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$

at (a,b)

(c) f has a <u>saddle point</u> @ (a,b) if

$f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a,b)

(d) the test is <u>inconclusive</u> if

$f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a,b).

Rmks: ① $f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$ is

called the <u>discriminant</u> or <u>Hessian</u> of f

(2) It seems like f_{xx} is playing a special role, as opposed to f_{yy}.

But in fact, if $f_{xx} f_{yy} - f_{xy}^2 > 0$ then

$$f_{xx} f_{yy} > f_{xy}^2 \geq 0$$

since a square is always ≥ 0

So $f_{xx} \leq 0$ if and only if $f_{yy} \leq 0$
and $f_{xx} \geq 0$ if and only if $f_{yy} \geq 0$.

So could have used f_{yy} instead of f_{xx} in parts (a) and (b).

Going back to our earlier example,

$$f(x,y) = x^2 + 2xy$$

$f_x = 2x + 2y \qquad f_y = 2x$

$f_{xx} = 2, \quad f_{xy} = 2, \quad f_{yy} = 0$

$$\Rightarrow \mathrm{Hess}(f) = \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = -4$$

$\Rightarrow$ saddle point @ $(0,0)$
(as we already observed).

Let's see some more interesting examples.

ex 2

$$f(x,y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$$

$$f_x = 2x - 2y - 2, \quad f_y = -2x + 4y + 2$$

$$f_{xx} = 2, \quad f_{xy} = -2, \quad f_{yy} = 4$$

Find critical points:

$$\left.\begin{aligned} f_x = 0 &\Rightarrow x - y = 1 \\ f_y = 0 &\Rightarrow x - 2y = 1 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 1, \\ y &= 0 \end{aligned}$$

$$f_{xx} = 2 > 0, \quad \mathrm{Hess}(f) = \begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} = 4 > 0$$

$(f(1,0) = 0)$

$\Rightarrow f$ has a local min @ $(1,0)$.

Could also have seen this by observing that $f(x,y) = (x-(y+1))^2 + y^2$. But above gives systematic method.

ex 3) $f(x,y) = e^y - ye^x$

$f_x = -ye^x, \quad f_y = e^y - e^x$

$f_{xx} = -ye^x, \quad f_{xy} = -e^x, \quad f_{yy} = e^y$

Find critical points:

$$\left.\begin{array}{l} f_x = 0 \Rightarrow y = 0 \\ f_y = 0 \Rightarrow x = y \end{array}\right\} \Rightarrow (x,y) = (0,0)$$

is the only crit pt

$f_{xx}(0,0) = 0$, oh no! But $f_{yy}(0,0) = 1$, so use that.

$$\text{Hess}(f) = \begin{vmatrix} -ye^x & -e^x \\ -e^x & e^y \end{vmatrix} = -1 \text{ @ } (0,0)$$

$\Rightarrow$ f has saddle point @ $(0,0)$.

$\boxed{\text{ex } 4}$ $f(x,y) = \ln(x+y) + x^2 - y$

$$f_x = \frac{1}{x+y} + 2x, \quad f_y = \frac{1}{x+y} - 1$$

$$f_{xx} = -\frac{1}{(x+y)^2} + 2, \quad f_{xy} = -\frac{1}{(x+y)^2},$$

$$f_{yy} = -\frac{1}{(x+y)^2}$$

f_x, f_y are undefined whenever $x+y=0$
but then so is f (can't take $\ln(0)$), so
don't need to include these as crit pts

$$\left.\begin{array}{l} f_y = 0 \Rightarrow x+y=1 \\ f_x = 0 \Rightarrow \dfrac{1}{x+y} + 2x = 0 \end{array}\right\} \Rightarrow \frac{1}{1} + 2x = 0$$

$$\Rightarrow x = -\frac{1}{2},$$

$$y = \frac{3}{2}$$

$\Rightarrow$ critical point @ $(-1/2, 3/2)$.

$f_{xx}(-1/2, 3/2) = 1$

$f_{xy}(-1/2, 3/2) = -1$

$f_{yy}(-1/2, 3/2) = -1$

$$\Rightarrow \mathrm{Hess}(f)_{(0,0)} = \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} = -2 < 0$$

$\Rightarrow f$ has a saddle point @ $(-1/2, 3/2)$

Recall: Extreme Value Theorem (for higher dimensions) tells us f always achieves an absolute max and min on any closed and bounded (= compact) region

So to find <u>absolute</u> extrema, look @ <u>boundary</u> points of the domain as well as critical points in the <u>interior</u> of the domain.

<u>ex 1</u> Find the absolute extrema of

$$D(x,y) = x^2 - xy + y^2 + 1$$

on the closed triangular plate in the first quadrant bounded by

$x=0,\ y=4,\ x=y$

① Find partials:

$D_x = 2x - y\ ,\ D_y = -x + 2y$

$D_{xx} = 2,\ D_{xy} = -1,\ D_{yy} = 2$

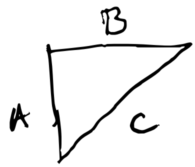
② Find critical points:

$$\left.\begin{array}{l} D_x = 0 \Rightarrow 2x - y = 0 \\ D_y = 0 \Rightarrow -x + 2y = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = 0, \\ y = 0 \end{array}$$

(not in interior)

③ Analyze boundary points:

We'll consider each side of the triangle



separately.

Ⓐ $x=0,\ 0 \le y \le 4,\ D = y^2 + 1$ min value 1

max value 17

Ⓑ $0 \le x \le 4,\ y = 4,\ D = x^2 - 4x + 17$

$$\left.\begin{array}{l} D_x = 2x - 4 \\ D_{xx} = 2 \end{array}\right\} \Rightarrow \text{local min } \underline{\text{on this segment}}$$

$D(2,4) = 13$

boundary pts of this segment: $D(0,4) = 17$

$D(4,4) = 17$

(c) $0 \le x = y \le 4$, $D = x^2 + 1$

min value: $D(0,0) = 1$

max value: $D(4,4) = 17$

Altogether, we see that the min value of D on this region is $D(0,0)=1$ and the max value is $D(4,4) = 17$.

ex 2) Find the extreme values of

$$f(x,y) = 48xy - 32x^3 - 24y^2$$

on the unit square $0 \le x \le 1$, $0 \le y \le 1$.

(1) Find partials:

$f_x = 48y - 96x^2$, $f_y = 48x - 48y$

$f_{xx} = -192x$, $f_{xy} = 48$, $f_{yy} = -48$

② Find critical points:

$$\left.\begin{array}{l} f_x = 0 \Rightarrow 48y - 96x^2 = 0 \\ f_y = 0 \Rightarrow 48x - 48y = 0 \end{array}\right\} \begin{array}{l} \Rightarrow x = y \\ \Rightarrow 48x - 96x^2 = 0 \\ \Rightarrow x - 2x^2 = 0 \\ \Rightarrow (x,y) = (0,0) \\ \quad \text{or } (1/2, 1/2) \end{array}$$

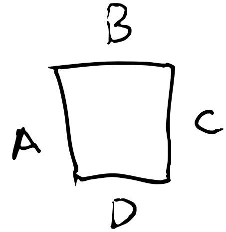
(0,0) is not in interior tho

$$\mathrm{Hess}(f)_{(1/2,1/2)} = \begin{vmatrix} -96 & 48 \\ 48 & -48 \end{vmatrix} = 48^2 > 0,$$

$f_{xx}(1/2,1/2) = -96 < 0$

so local max @ $(1/2, 1/2)$, $f(1/2,1/2) = 2$

③ Analyze boundary points:



Ⓐ $x = 0,\ 0 \le y \le 1 \Rightarrow f = -24y^2$

max val $= 0$, min val $= -24$

Ⓑ $0 \le x \le 1,\ y = 1 \Rightarrow f = 48x - 32x^3 - 24$

$f_x = 48 - 96x^2$

$f_x = 0 \Rightarrow 2x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{2}}$ (since $-\frac{1}{\sqrt{2}}$ is not in our domain)

$$f_{xx}(1/\sqrt{2}, 1) = \frac{-192}{\sqrt{2}}$$

$\Rightarrow$ f has local max on <u>this segment</u>

$$f(1/\sqrt{2}, 1) = \frac{48}{\sqrt{2}} - \frac{32}{2\sqrt{2}} - 24$$

$$= 16\sqrt{2} - 24$$

@ boundary pts of this segment,

$f(0,1) = -24$, $f(1,1) = -8$

(c) $x=1$, $0 \le y \le 1 \Rightarrow f = -24y^2 + 48y - 32$

$f_y = -48y + 48$

$f(1,0) = -32$

$f(1,1) = -8$

(D) $0 \le x \le 1$, $y = 0 \Rightarrow f = -32x^3$

max value $f(0,0) = 0$, min value $f(1,0) = -32$

Altogether, we see that the absolute max/min of f on the unit square are

max: $f(1/2, 1/2) = 2$

min: $f(1,0) = -32$.