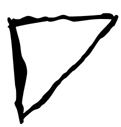
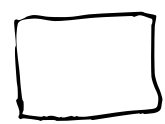
# §14.8 Lagrange Multipliers

In last lecture, saw some examples of maximizing/minimizing a function <u>subject to a constraint</u>

e.g. restricted to  or 

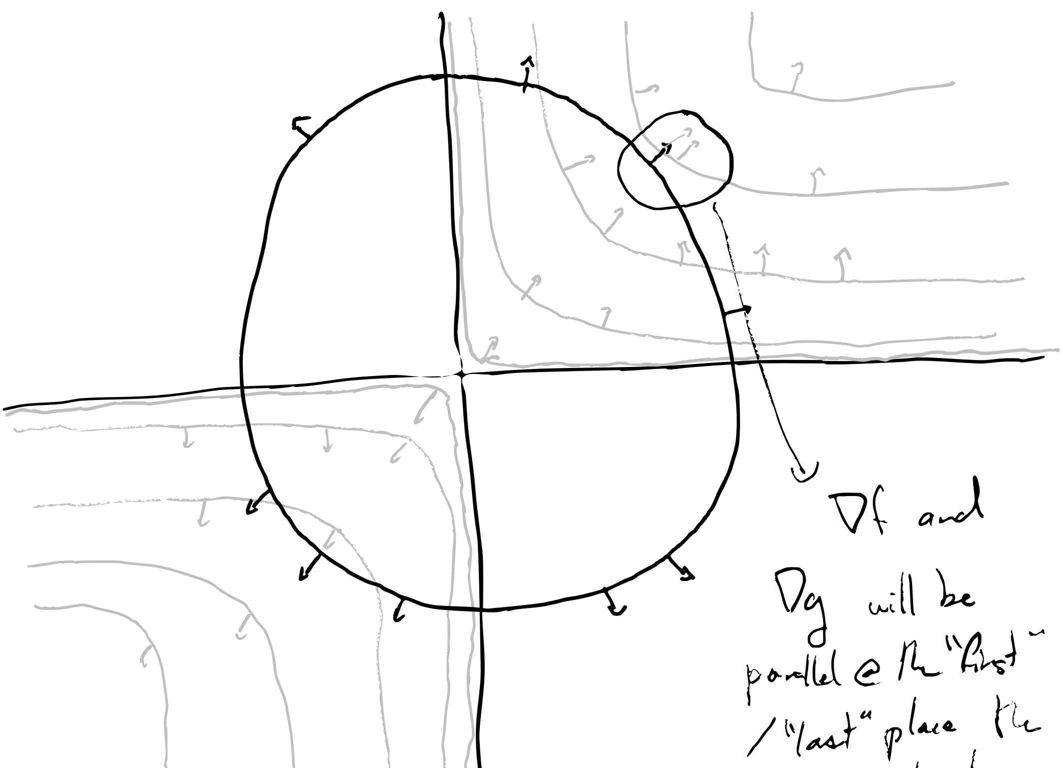
In these cases we could analyze the boundary "by hand". This won't be feasible for more complicated constraints though.

Want a general method to solve:

> What are the maximum/minimum values of a function $f(x,y,z)$ <u>subject to the constraint</u> $g(x,y,z)=0$ ?

ex Find the extreme values of $f(x,y) = xy$ subject to the constraint $x^2+y^2=25$.

The level curves of $f$ are hyperbolas; whereas $x^2+y^2=25$ defines a circle of radius 5.



$\nabla f$ and $\nabla g$ will be parallel @ the "first" / "last" place the curves touch

Another way of saying what we're doing is: find $c$ as large/low as possible such that the level curve $\{f(x,y)=c\}$ intersects the constraint domain $\{x^2+y^2-25=0\}$.

If you imagine increasing the value of $c$ as far as possible until the level curve "breaks away" from the constraint curve — or alternatively making $c$ too large, then decreasing it just until the curves touch — then we can see visually that the gradients of $f(x,y)=xy$ and of $g(x,y)=x^2+y^2-25$ will be parallel.

Here is a rigorous justification for this.

<u>Thm</u> Let $f(x,y)$ be a diff'ble fn of two variables, let

$$C: \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

be a curve. If $(x_0, y_0) = \vec{r}(t_0)$ is a local extremum for $f$ <u>relative to the curve $C$</u>, then

$$\nabla f_{(x_0,y_0)} \cdot \vec{r}\,'(t_0) = 0.$$

<u>Pf</u> The hypothesis is exactly saying that the function <u>of one variable</u>

$$p(t) = f(\vec{r}(t)) = f(x(t), y(t))$$

has a local extremum @ $t_0$. We know from one-variable calculus that $p'(t_0) = 0$.

But the chain rule gives

$$p'(t) = Df_{\vec{r}(t)} \cdot \vec{r}\,'(t)$$

$\square$

We also know the gradient of $g$ is orthogonal to the level curve which is the range of $\vec{r}(t)$. $\{g=0\}$, which is the range of $\vec{r}(t)$.
Since $Df_{(x_0,y_0)}$ and $Dg_{(x_0,y_0)}$ are both orthogonal to $\vec{r}\,'(t_0)$, they must be parallel to one another, assuming $Dg$ is not $0$.

(In three dimensions or higher, we apply the theorem to <u>every</u> curve in the surface passing through the given point to see that $Df$ is parallel to $Dg$).

So:

$$f(x,y) = xy \Rightarrow \nabla f = y\vec{i} + x\vec{j}$$

$$g(x,y) = x^2+y^2-25 \Rightarrow \nabla g = 2x\vec{i} + 2y\vec{j}$$

Want these to be parallel!

$$\nabla f = \lambda \nabla g$$

$$\underbrace{y\vec{i}} + x\vec{j} = \underbrace{2\lambda x\vec{i}} + 2\lambda y\vec{j}$$

gives $\lambda = \dfrac{y}{2x}$

$$x = 2\left(\frac{y}{2x}\right)y \Rightarrow \boxed{x^2 = y^2}$$

So the max/min values will be when
$x^2 = y^2$ (and $x^2+y^2=25$ still)

$\Rightarrow (x,y) = \left(\pm \frac{5}{\sqrt{2}}, \pm \frac{5}{\sqrt{2}}\right)$

$\Rightarrow$ max value is $\frac{25}{\sqrt{2}}$, min value is $-\frac{25}{\sqrt{2}}$.

To summarise:

**Method of Lagrange Multipliers**

Let $f(x,y,z)$ and $g(x,y,z)$ be differentiable, suppose $\nabla g \neq 0$ whenever $g(x,y,z)=0$.

The extreme values of $f$ subject to the constraint $g(x,y,z)=0$, if they exist, are located @ points where

$$\nabla f_{(x,y,z)} = \lambda \nabla g_{(x,y,z)}.$$

($\lambda$ is called the Lagrange multiplier)

ex Find the point on the curve

$$x^2 y = 2$$

closest to the origin.

Soln Trying to minimize $f(x,y) = x^2 + y^2$
subject to $g(x,y) = x^2 y - 2 = 0$.

$\nabla f = 2x\vec{i} + 2y\vec{j}$

$\nabla g = 2xy\vec{i} + x^2\vec{j}$

($x \neq 0$ since won't be on $x^2 y = 2$)

$\nabla f = \lambda \nabla g \Rightarrow$ $2x = 2\lambda xy \Rightarrow \lambda = 1/y$

$2y = \lambda x^2 \Rightarrow 2y^2 = x^2$

Put $2y^2 = x^2$ into $x^2 y = 2$ to get
$2y^3 = 2 \Rightarrow y = 1$ and $x = \sqrt{2}$.

So $(\sqrt{2},1)$ is the closest point to the origin on $\{x^2y=2\}$.

ex Find the point(s) on

$$xyz=1$$

closest to the origin.

Soln Trying to minimize $f(x,y,z)=x^2+y^2+z^2$ subject to $g(x,y,z)=xyz=1$.

$$\nabla f=2x\vec{i}+2y\vec{j}+2z\vec{k}$$

$$\nabla g=yz\vec{i}+xz\vec{j}+xy\vec{k}$$

$$\nabla f=\lambda\nabla g \Rightarrow \begin{array}{l} 2x=\lambda yz \\ 2y=\lambda xz \\ 2z=\lambda xy \end{array}$$

$2x = \lambda yz$ gives $\lambda = \dfrac{2x}{yz}$.

Then we get

$$2y = \lambda xz \Rightarrow 2y = \frac{2x}{yz} xz \Rightarrow x^2 = y^2$$

$$2z = \lambda xy \Rightarrow 2z = \frac{2x}{yz} xy \Rightarrow z^2 = x^2$$

So $x, y, z$ are all $\pm r$ for some $r \geq 0$. But $xyz = 1 \Rightarrow r = 1$. So the points closest to the origin are $(1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$, and $(1,-1,-1)$.

$\lfloor$

ex Suppose the Celsius temperature of a point on the sphere

$$x^2+y^2+z^2=1$$

is given by

$$T=20xyz^2.$$

Find the point(s) of highest temperature.

Soln Trying to maximize/minimize

$$T(x,y,z)=20xyz^2$$

subject to

$$g(x,y,z)=x^2+y^2+z^2-1=0$$

$$\nabla T = 20yz^2\vec{i} + 20xz^2\vec{j} + 40xyz\vec{k}$$

$$\nabla g = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{aligned} 20yz^2 &= \lambda 2x \\ 20xz^2 &= \lambda 2y \\ 40xyz &= \lambda 2z \end{aligned}$$

$$\lambda = \frac{10yz^2}{x} \Rightarrow \begin{aligned} 20xz^2 &= \left(\frac{10yz^2}{x}\right)(2y) \\ 40xyz &= \left(\frac{10yz^2}{x}\right)(2z) \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} x^2z^2 &= y^2z^2 \\ 2x^2yz &= yz^3 \end{aligned}\right] \Rightarrow x^2 = y^2, \quad 2x^2 = z^2$$

$$\Rightarrow x = \pm r, \quad y = \pm r, \quad z = \pm r\sqrt{2}$$

$x^2+y^2+z^2=1 \Rightarrow r=\sqrt{1/4}=1/2$

So points of highest/lowest temperature are among $\left(\pm 1/2, \pm 1/2, \pm \frac{\sqrt{2}}{2}\right)$

Highest temp = $5/2$

Lowest temp = $-5/2$

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<u>Multiple constraints</u>

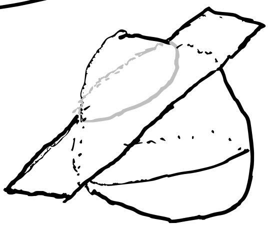
What about finding extrema of $f$ subject to <u>multiple</u> constraints

$$g_1(x,y,z)=0 \qquad ?$$
$$g_2(x,y,z)=0$$

The intersection of two surfaces $\{g_1=0\}$ and $\{g_2=0\}$ will usually be a curve.

This is always the case if $\nabla g_1$ and $\nabla g_2$ are not parallel.

ex



The sphere $\{x^2+y^2+z^2-1=0\}$ intersected with the plane $\{-x+z-1=0\}$ is a circle (in particular, a curve).

Note that $\nabla(x^2+y^2+z^2-1) = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$ is not parallel to $\nabla(-x+z) = -\vec{i} + \vec{k}$ at any point of intersection:

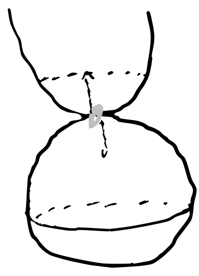
$$2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \lambda(-\vec{i} + \vec{k})$$

$$\Rightarrow \quad y=0, \quad x=-z$$

$$-x+z-1=0 \Rightarrow 2z-1=0 \Rightarrow z=\frac{1}{2}, \quad x=-\frac{1}{2}$$

but $(-1/2, 0, 1/2) \notin \{x^2+y^2+z^2-1=0\}$.

ex



The intersection of the paraboloid $\{x^2+y^2-z=0\}$ with the sphere $\{x^2+y^2+(z+1)^2-1=0\}$ is a single point (the origin), rather than a curve. Note that at this point of intersection,

$$\nabla(x^2+y^2-z) = 2x\vec{i} + 2y\vec{j} - \vec{k} \underset{(0,0,0)}{=} -\vec{k}$$

and

$$\nabla(x^2+y^2+(z+1)^2-1) = 2x\vec{i} + 2y\vec{j} + (2z+2)\vec{k} \underset{(0,0,0)}{=} 2\vec{k}$$

are parallel.

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So we'll assume $\nabla g_1$ and $\nabla g_2$ are never parallel on the intersection $\{g_1=0\} \cap \{g_2=0\}$.

As before, if $f$ has a local extremum @ $(x_0, y_0, z_0)$ subject to the constraints $\{g_1 = 0\}$ and $\{g_2 = 0\}$, then let $\vec{r}(t)$ be a curve landing in $\{g_1 = 0\} \cap \{g_2 = 0\}$ with $\vec{r}(t) = (x_0, y_0, z_0)$. Then the function of one variable $f(\vec{r}(t))$ has a local extremum at $0$, so

$$\Big[ f(\vec{r}(t)) \Big]'_{t=0} = \nabla f(x_0, y_0, z_0) \cdot \vec{r}\,'(0) = 0$$

So $\nabla f$ will be orthogonal to the velocity vector $\vec{r}\,'(0)$. We also know that $\vec{r}\,'(0)$ will be orthogonal to both the gradients $\nabla g_1(x_0, y_0, z_0)$ and $\nabla g_2(x_0, y_0, z_0)$. It follows that $\nabla f$ is a linear combination of $\nabla g_1$ and $\nabla g_2$ at constrained local extrema, i.e.

$$\boxed{\nabla f = \lambda \nabla g_1 + \mu \nabla g_2}$$

ex) The plane $x - y + z = 0$ cuts the cylinder $y^2 + z^2 = 1$ in an ellipse. Find the points on the ellipse closest to and farthest from the origin.

soln) Trying to optimize

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to

$$g_1(x,y,z) = x - y + z = 0$$

$$g_2(x,y,z) = y^2 + z^2 - 1 = 0$$

$$\nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla g_1 = \vec{i} - \vec{j} + \vec{k}$$

$$\nabla g_2 = 2y\vec{j} + 2z\vec{k}$$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$2x\vec{i} + 2y\vec{j} + 2z\vec{k} = \lambda(\vec{i} - \vec{j} + \vec{k})$$
$$+ \mu(2y\vec{j} + 2z\vec{k})$$

$$\Rightarrow \lambda = 2x$$

$$\Rightarrow 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = 2x\vec{i} + (2\mu y - 2x)\vec{j}$$
$$+ (2\mu z + 2x)\vec{k}$$

$$\left.\begin{array}{l} 2y = 2\mu y - 2x \Rightarrow \mu = 1 + x/y \\ 2z = 2\mu z + 2x \Rightarrow \mu = 1 - x/z \end{array}\right\} \Rightarrow x = 0 \text{ or } y = -z$$

If $x = 0$, $x - y + z = 0 \Rightarrow y = z$

$y^2 + z^2 = 1 \Rightarrow (x, y, z) = (0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$.

If $y = -z$, $y^2 + z^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{2}}, z = \mp\frac{1}{\sqrt{2}}$

$x - y + z = 0$

$\Rightarrow x = 2y = \pm\sqrt{2}$

So looking at points

$$\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),$$

$$\left(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

The top two are the closest to the origin (distance 1), the bottom two are the farthest (distance $\sqrt{3}$).