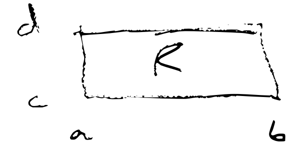
§15.1 Double and Iterated Integrals over Rectangles

Previous chapters were about <u>differential</u> calculus in multiple variables.

Now for <u>integral</u> calculus.

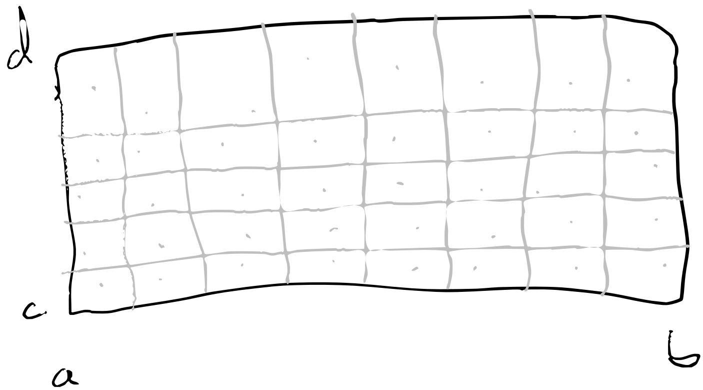
Let $f(x,y)$ be a function of two variables, and let



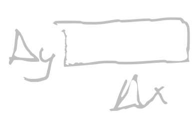
$$R = [a,b] \times [c,d]$$
$$= \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$$

be a rectangle in the plane.

To "integrate" f, we partition the x-axis <u>and</u> the y-axis and sample f inside the grid.



If a subrectangle has length Δx and height Δy, its area is $\Delta A = \Delta x \, \Delta y$.



The <u>Riemann sum</u> associated to a partition (and choice of sample points) is

$$S = \sum_{\text{rectangles}} f(x_k, y_k)\, \Delta A_k$$

(x_k, y_k) is point sampled from k^{th} rectangle

This involves a lot of choices: the choice of partition, and the choice of sample points.

Define the *norm* $\|P\|$ of a partition P to be the max of the lengths and widths of the rectangles in the partition P. If the limit

$$\lim_{\|P\| \to 0} \sum_{k \text{ in } P} f(x_k, y_k) \Delta A_k$$

exists, we say that f is *integrable* and denote the limit by

$$\iint_R f(x,y)\, dA \quad \text{or} \quad \iint_R f(x,y)\, dx\, dy.$$

To be more explicit, we could take

$$\lim_{m,n\to\infty} \frac{(b-a)(d-c)}{mn} \sum_{i=0}^{m} \sum_{j=0}^{n} \left[f\left(a+\frac{i}{m}(b-a),\ c+\frac{j}{n}(d-c)\right)\right]$$

to calculate $\iint_R f(x,y)\,dA$.

If we graph $z=f(x,y)$, then $\iint_R f(x,y)\,dA$ represents the (signed) <u>volume</u> of the surface over R.

This was just to <u>define</u> $\iint_R f(x,y)\,dA$ - how do we <u>calculate</u> it?

Fubini's thm

If $f(x,y)$ is continuous on the rectangle $R = \{a \le x \le b,\ c \le y \le d\}$ then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

integrate wrt x first

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

integrate wrt y first

so we can calculate a <u>double integral</u> as an <u>iterated integral</u>. In particular, we can integrate in <u>either order</u>.

ex 1 Find

$$\int_{\pi}^{2\pi}\int_{0}^{\pi} (\sin x + \cos y)\,dx\,dy$$

soln

$$\int_{\pi}^{2\pi}\int_{0}^{\pi} (\sin x + \cos y)\,dx\,dy$$

integrate treating y as a constant

$$= \int_{\pi}^{2\pi} \Big[-\cos x + x\cos y\Big]_0^{\pi}\,dy$$

$$= \int_{\pi}^{2\pi} \big[2 + \pi\cos y\big]\,dy$$

$$= \Big[2y + \pi\sin y\Big]_{\pi}^{2\pi} = 2\pi$$

If we integrated in the opposite order, it would be

$$\int_0^{\pi}\int_{\pi}^{2\pi}\left[\sin(x) + \cos(y)\right] dy\, dx$$

$$= \int_0^{\pi}\left[y\sin(x) + \sin(y)\right]_{\pi}^{2\pi} dx$$

$$= \int_0^{\pi} \pi\sin(x)\, dx = \left[-\pi\cos(x)\right]_0^{\pi}$$

$$= 2\pi.$$

<u>ex 2</u> Find

$$\iint_R y\sin(x+y)\, dA \qquad R: -\pi \le x \le 0,\ 0 \le y \le \pi$$

soln

$$\iint_R y\sin(x+y)\,dA = \int_0^{\pi}\int_{-\pi}^{0} y\sin(x+y)\,dx\,dy$$

$$= \int_0^{\pi}\Big[-y\cos(x+y)\Big]_{-\pi}^{0}\,dy$$

$$= \int_0^{\pi}\Big[-y\cos(y) + y\cos(y-\pi)\Big]\,dy$$

$$= \int_0^{\pi} -2y\cos(y)\,dy$$

integrate by parts

$$= -2y\sin(y)\Big]_0^{\pi} + \int_0^{\pi} 2\sin(y)\,dy$$

$$= \Big[-2\cos(y)\Big]_0^{\pi} = 4.$$

In the other order,

$$\iint_R y \sin(x+y)\, dA = \int_{-\pi}^{0}\int_0^{\pi} y \sin(x+y)\, dy\, dx$$

(integration by parts)

$$= \int_{-\pi}^{0}\left(\Big[-y\cos(x+y)\Big]_0^{\pi} + \int_0^{\pi} \cos(x+y)\, dy\right) dx$$

$$= \int_{-\pi}^{0}\left(\pi\cos(x) + \Big[\sin(x+y)\Big]_0^{\pi}\right) dx$$

$$= \int_{-\pi}^{0}\big(\pi\cos(x) - 2\sin(x)\big)\, dx = \Big[\pi\sin(x) + 2\cos(x)\Big]_{-\pi}^{0}$$

$$= 4.$$

ex 3 Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle $0 \le x \le 1,\ 0 \le y \le 2$.

soln

$$V = \int_0^2 \int_0^1 (4 - y^2)\,dx\,dy$$

$$= \int_0^2 (4 - y^2)\,dy = \left[4y - \frac{1}{3}y^3\right]_0^2$$

$$= \frac{16}{3}$$