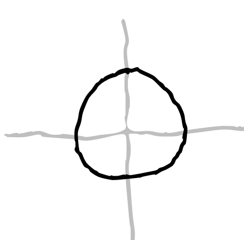
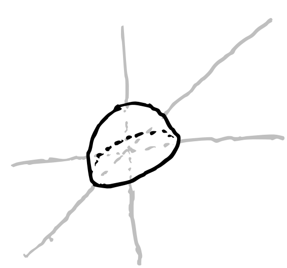
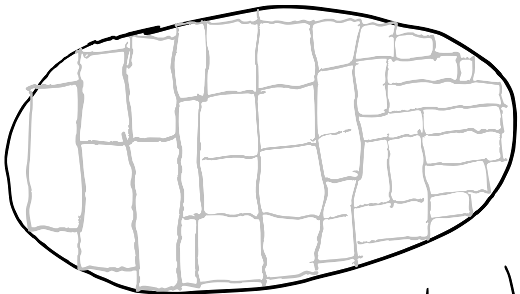
§15.2 Double Integrals over General Regions

How can we integrate over more general shapes? For instance, if $f(x,y) = \sqrt{1-x^2-y^2}$, then the integral of f over the unit circle in the plane will give the <u>volume</u> of the upper half-unit square.





To integrate over a region R, we try to fill it with rectangles.



To such a set of rectangles, we associate a Riemann sum

height of k^{th} rectangle

$$\sum_{\text{rectangles}} f(x_k, y_k)\,\Delta x_k\,\Delta y_k$$

width of k^{th} rectangle

point sampled from k^{th} rectangle

As we make the norm of the partition smaller (i.e. let the width <u>and</u> height of the rectangles go to 0), we can approximate the shape of R better and better. In the cases we care about, this will converge to a limit, which we call the integral of f over R,

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy.$$

This satisfies the following properties:

(1) Linearity: if c is a constant, then

$$\iint_R c f(x,y)\,dA = c\iint_R f(x,y)\,dA$$

and

$$\iint_R (f(x,y)+g(x,y))\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

(2) Domination: if $f \geq g$ on R, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

(3) Additivity: if R is the union of two non-overlapping subsets R_1, R_2 (in symbols: $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$),

then

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

To calculate these, use more general form of <u>Fubini's theorem</u>:

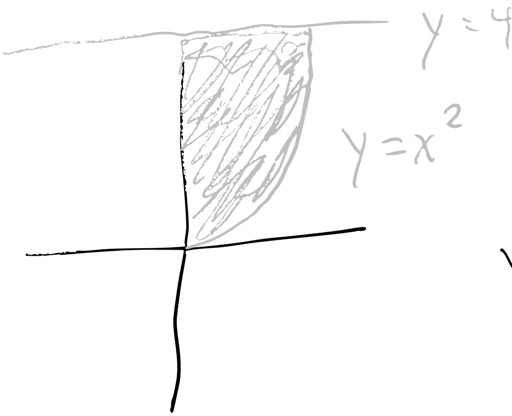
<u>Thm</u> (a) If R is given by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

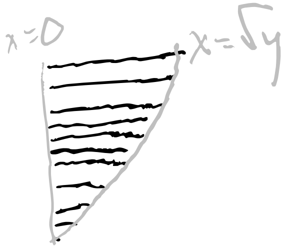
(b) If R is given by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

ex 1] Let R be the region enclosed by the graph of $y=x^2$, $y=4$, $y=0$, and $x=0$, as shown here.

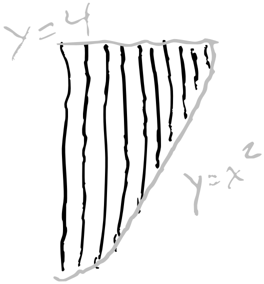


If we integrate with "horizontal slices", i.e. with respect to x first, our integral will be



$$\int_0^y \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$$

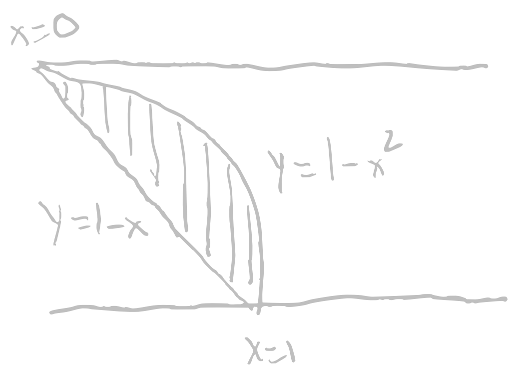
If we integrate with "vertical slices", i.e. with respect to y first, it will be



$$\int_0^2\int_{x^2}^4 f(x,y)\,dy\,dx$$

<u>ex 2</u> Rewrite $\displaystyle\int_0^1\int_{1-x}^{x^2} f(x,y)\,dy\,dx$ with the opposite order of integration.

Sketch region.



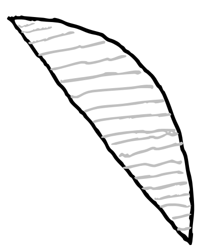
We see that $0 \le y \le 1$. Moreover,

$$y = 1 - x \Rightarrow x = 1 - y$$

$$y = 1 - x^2 \Rightarrow x = \sqrt{1-y}$$

So

$$\int_0^1 \int_{1-x}^{1-x^2} f(x,y)\, dy\, dx = \int_0^1 \int_{1-y}^{\sqrt{1-y}} f(x,y)\, dx\, dy$$



Examples

(1) $$\int_0^1 \int_{1-x}^{1-x^2} 2y \, dy \, dx$$

$$= \int_0^1 y^2 \Big]_{1-x}^{1-x^2} dx$$

$$= \int_0^1 \left[(1-x^2)^2 - (1-x)^2\right) dx$$

$$(1 - 2x^2 + x^4)$$

$$- (1 - 2x + x^2)$$

$$= \int_0^1 (x^4 - 3x^2 + 2x) \, dx$$

$$= \frac{1}{5} - 1 + 1 = \frac{1}{5}$$

In the other order,

$$\int_0^1 \int_{1-y}^{\sqrt{1-y}} 2y \, dx \, dy$$

$$= \int_0^1 2y\sqrt{1-y} \, dy - \underbrace{\int_0^1 2y(1-y) \, dy}_{= 1 - \frac{2}{3} = \frac{1}{3}}$$

$u = y \quad dv = \sqrt{1-y}\,dy$

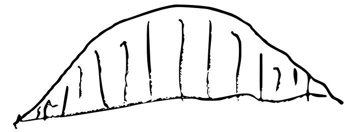
$du = dy \quad v = -\frac{2}{3}(1-y)^{3/2}$

$$= \cancel{-\frac{4}{3} y(1-y)^{3/2} \Big]_0^1} + \int_0^1 \frac{4}{3}(1-y)^{3/2} \, dy - \frac{1}{3}$$

$$= \frac{4}{3} \cdot \frac{2}{5} - \frac{1}{3} = \frac{1}{5}$$

ex 2

$$\int_0^{\pi}\int_0^{\sin x} y \, dy \, dx$$



$$= \int_0^{\pi} \left[y^2/2 \right]_0^{\sin x} dx = \int_0^{\pi} \frac{\sin^2 x}{2} dx$$

integrating even function

$$= \int_0^{\pi/2} \sin^2 x \, dx$$

#65 in Table of Integrals @ back of book

$$= \left[\frac{x}{2} - \frac{\sin 2x}{4} \right]_0^{\pi/2} = \frac{\pi}{4}$$

In other order,

$$\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \int_0^{1}\int_{\sin^{-1}y}^{\pi-\sin^{-1}y} y\,dx\,dy$$

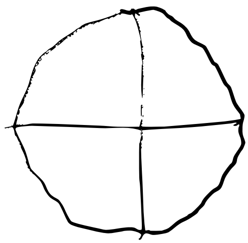
$$= \int_0^1 \left(\pi y - 2y\sin^{-1}(y)\right) dy$$

= give up. Sometimes one order of integration is much easier than the other!

If you really wanted to,

$$\int 2y\sin^{-1}(y)\,dy = y^2\sin^{-1}(y) - \frac{1}{2}\sin^{-1}(y) + \frac{1}{2}y\sqrt{1-y^2} + C$$

ex 3 Let's calculate the volume of the upper half-sphere.



$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\, dx\, dy$$

Use formula (#45 in Table of Integrals @ back of book)

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

with $a = \sqrt{1-y^2}$

$$\hookrightarrow = \int_{-1}^{1} \frac{x}{2}\sqrt{1-x^2-y^2} + \frac{1-y^2}{2}\sin^{-1}\frac{x}{\sqrt{1-y^2}}\Bigg]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$$

$$= \int_{-1}^{1}\left(0 + \frac{1-y^2}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right)dy$$

$$= \frac{\pi}{2}\int_{-1}^{1}(1-y^2)\,dy$$

$$= \frac{\pi}{2}\left[1-\frac{1}{3}+1-\frac{1}{3}\right] = \frac{2\pi}{3}$$

(so the unit sphere has volume $\frac{4\pi}{3}$)