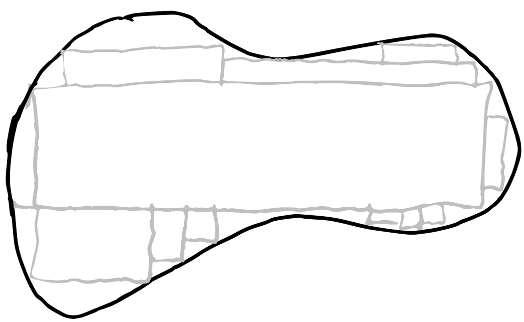
§15.3 Area by double integration

If we integrate the constant function $f(x,y)=1$ over a region R,



the Riemann sums $\sum f(x_k,y_k)\Delta A_k$ become the sum $\sum \Delta A_k$ of the areas of the rectangles. This agrees with what we intuitively think of the area of R.

<u>Def</u> The <u>area</u> of a closed and bounded plane region R is

$$\iint_R dA.$$

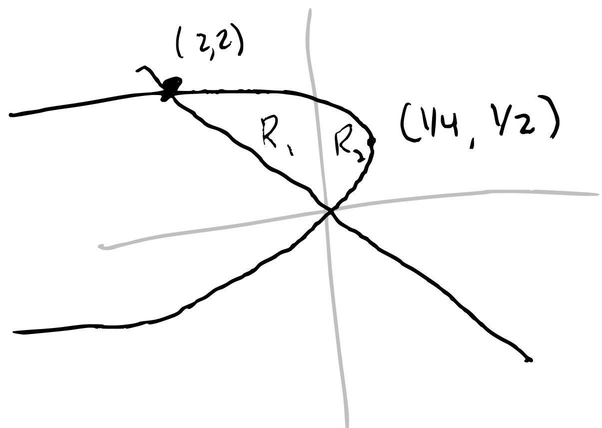
<u>ex 1</u> Find the area enclosed by the curves $x = y - y^2$ and $y = -x$.

First, let's find the points of intersection:

$y=-x \Rightarrow x=-y$ & $x=y-y^2$

$\Rightarrow -y = y-y^2 \Rightarrow y=0,\ 2$

so intersect @ $(0,0)$ and $(-2,2)$



a) If we integrate with vertical slices,

$$x=y-y^2 \Rightarrow y=\frac{1\pm\sqrt{1-4x}}{2}$$

Since the "lower value" for y changes @ $x=0$, we'll split up the region

$$\iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA$$

$$\iint_{R_1} dA = \int_{-2}^{0} \int_{-x}^{\frac{1+\sqrt{1-4x}}{2}} dy\, dx$$

$$\iint_{R_2} dA = \int_{0}^{1/4} \int_{\frac{1-\sqrt{1-4x}}{2}}^{\frac{1+\sqrt{1-4x}}{2}} dy\, dx$$

This is pretty complicated! Let's try it the other way.

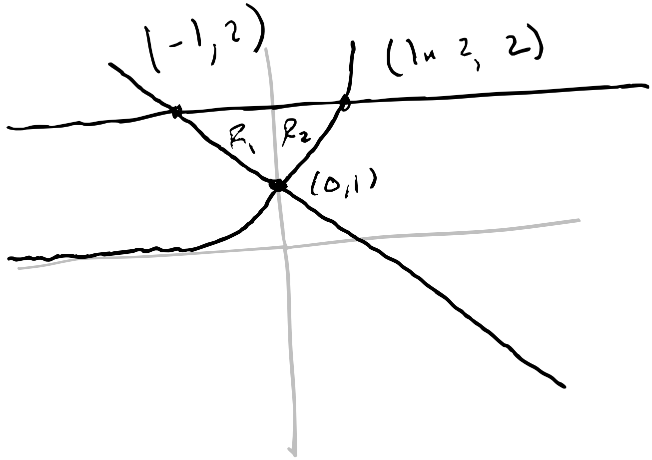
b) With horizontal slices,

$$\iint_R dA = \int_0^2 \int_{-y}^{y-y^2} dx\, dy$$

$$= \int_0^2 \left(2y - y^2\right) dy$$

$$= y^2 - \frac{y^3}{3} \Bigg]_0^2 = \frac{4}{3}$$

ex 2) Find the area of the region enclosed by $y=1-x$, $y=2$, and $y=e^x$.



a) With vertical slices!

$$\iint_R dA = \underbrace{\iint_{R_1} dA}_{=\frac{1}{2} \text{ by geometry}} + \iint_{R_2} dA$$

$$\iint_{R_2} dA = \int_0^{\ln 2} \int_{e^x}^{2} dy\, dx$$

$$= \int_0^{\ln 2} (2 - e^x)\, dx$$

$$= \left[2x - e^x\right]_0^{\ln 2} = 2\ln(2) - 1$$

so $A = \frac{1}{2} + (2\ln(2) - 1) = 2\ln(2) - \frac{1}{2}$

b) with horizontal slices:

$$y = 1 - x \Rightarrow x = 1 - y$$

$$y = e^x \Rightarrow x = \ln(y)$$

$$\iint_R dA = \int_1^2 \int_{1-y}^{\ln(y)} dx\, dy$$

$$= \int_1^2 \ln(y) + y - 1$$

$$= y\ln(y) - y + \frac{y^2}{2} - y \Big]_1^2$$

$$= (2\ln(2) - 2 + 2 - 2) - (0 - 1 + \tfrac{1}{2} - 1) = 2\ln(2) - \frac{1}{2}$$

Average Values

If f is a constant function $f(x,y)=c$, then

$$\iint_R f(x,y)\,dA = c\,\text{Area}(R)$$

We define the average value of f on R as

$$\text{average value of } f \text{ on } R = \frac{\iint_R f(x,y)\,dA}{\iint_R dA}$$

i.e. $c = \operatorname{avg}(f)$ is such that

$$\iint_R f(x,y)\,dA = \iint_R c\,dA.$$

<u>ex 1</u> Find the average value of $f(x,y) = \sin(x+y)$ over the rectangle $0 \le x \le \frac{\pi}{2}$, $0 \le y \le \pi$.

$$\iint_R f(x,y)\,dA = \int_0^{\pi}\int_0^{\pi/2} \sin(x+y)\,dx\,dy$$

$$= \int_0^{\pi} -\cos(x+y)\Big]_0^{\pi/2}\,dy$$

$$\cos(\pi/2 + y) = -\sin(y), \quad \text{so}$$

$$= \int_0^{\pi} (\sin(y) + \cos(y))\, dy$$

$$= -\cos(\pi) + \sin(\pi) + \cos(0) - \sin(0) = 2$$

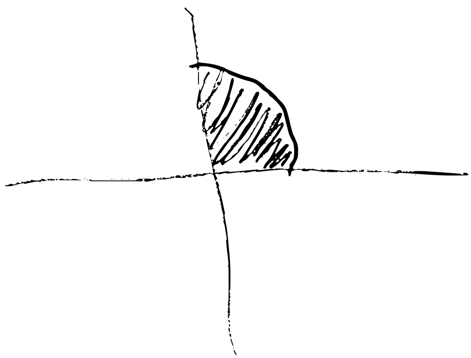
Area of rectangle is $\frac{\pi^2}{2}$

so average value is

$$\frac{2}{\pi^2/2} = \frac{4}{\pi^2}.$$

ex 2] Find the average value of $f(x,y) = 2xy$ on the quarter-circle

$\{x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0\}$



$$\iint_R f(x,y)\,dA = \int_0^1 \int_0^{\sqrt{1-y^2}} 2xy \; dx\, dy$$

$$= \int_0^1 x^2 y \Big]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 (1-y^2)y = \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{6}$$

Area of quarter-circle is $\frac{\pi}{4}$, so

$$\text{average value} = \frac{1/6}{\pi/4} = \frac{2}{3\pi}.$$

(If we used calculus to get the area of R, it would be

$$\iint_R dA = \int_0^1 \int_0^{\sqrt{1-y^2}} dx\, dy = \int_0^1 \sqrt{1-y^2}\, dy$$

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$$= \left[\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}(y)\right]_0^1 = \frac{\pi}{4}$$)