§15.6 Applications

Recall: if D is a solid region in space, with **density** $\delta(x,y,z)$ at the point (x,y,z), then the **mass** M of D is the integral of δ over D:

$$M = \iiint_D \delta(x,y,z)\,dV$$

An important related concept is the center of mass.

First: the first moment of D with respect to a plane P is

$$M_P = \iiint_D d_P(x,y,z)\,\delta(x,y,z)\,dV$$

where $d_P(x,y,z)$ is the distance from the point (x,y,z) to the plane P.

We'll only need this for the coordinate planes, so the formulas become:

$$M_{yz} = \iiint_D x\,\delta(x,y,z)\,dV$$

$$M_{xz} = \iiint_D y\,\delta(x,y,z)\,dV$$

$$M_{xy} = \iiint_D z\,\delta(x,y,z)\,dV$$

Now the <u>center of mass</u> of D is $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = M_{yz}/M$$

$$\bar{y} = M_{xz}/M$$

$$\bar{z} = M_{xy}/M$$

In physics/engineering, can often treat D as a point mass, i.e. mass M concentrated at the point $(\bar{x}, \bar{y}, \bar{z})$.

Motivation: a lever

is balanced when

$$m_1 d_1 = m_2 d_2 .$$

where m_i = mass of object, d_i = distance to fulcrum of lever.

So equivalently, $\bar{x}$ and $\bar{y}$ are such that

$$\iint_D (x-\bar{x})\,\delta\, dA = 0 \text{ and}$$

$$\iint_D (y-\bar{y})\,\delta\, dA = 0 .$$

For 2d regions, center of mass is defined the same way, except now we use coordinate axes instead of coordinate planes:

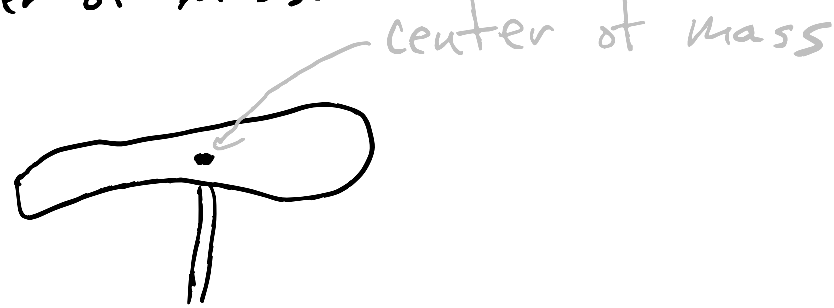
$$M = \iint_D \delta(x,y)\, dA \qquad \bar{x} = \frac{M_y}{M}$$

beware!

$$M_y = \iint_D x\, \delta(x,y)\, dA \qquad \bar{y} = \frac{M_x}{M}$$

$$M_x = \iint_D y\, \delta(x,y)\, dA$$

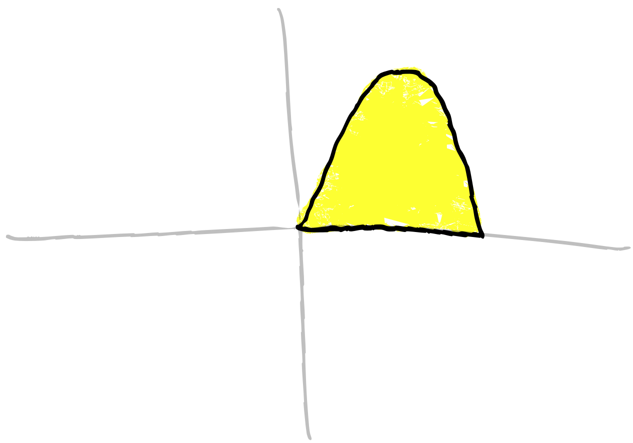
e.g. if you're trying to balance a plate on a thin column, you should place it below the center of mass.



When the density is 1, i.e. $\delta(x,y,z)=1$ for all (x,y,z), the center of mass is called the <u>centroid</u> of the region.

ex) Find the centroid of the region bounded by

$y = 0$ and $y = 2\sin(x)$ $0 \leq x \leq \pi$



Find area:

$$M = \int_0^\pi 2\sin(x)\,dx = \Big[-2\cos x\Big]_0^\pi$$

$$= 4$$

Find moments:

$$M_y = \int_0^{\pi} \int_0^{2\sin(x)} x \, dy \, dx$$

$$= \int_0^{\pi} 2x \sin(x) \, dx$$

$$= \left[-2x \cos(x) \right]_0^{\pi} + \int_0^{\pi} 2\cos(x) \, dx$$

$$= 2\pi + 0 + 0$$

$$= 2\pi$$

$$M_x = \int_0^{\pi}\int_0^{2\sin(x)} y \, dy \, dx$$

$$= \int_0^{\pi} 2\sin^2(x) \, dx$$

$$= \left[x - \frac{\sin(2x)}{2} \right]_0^{\pi} = \pi$$

So centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right) = \left(\frac{\pi}{2}, \frac{\pi}{4} \right).$$

Note we could have determined $\bar{x} = \frac{\pi}{2}$ just by symmetry.

ex] Find the center of mass of the square plate $\{0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{2}\}$ with density

$$\delta(x,y) = 2 + \cos(x+y)$$

Find mass:

$$M = \int_0^{\pi/2}\int_0^{\pi/2} (2 + \cos(x+y))\,dx\,dy$$

$$= \int_0^{\pi/2} (\pi + \sin(\pi/2 + y) - \sin(y))\,dy$$

$$= \left[\pi y - \cos\left(\frac{\pi}{2}+y\right) + \cos(y) \right]_0^{\pi/2}$$

$$= \frac{\pi^2}{2}$$

Find moments:

$$M_y = \int_0^{\pi/2}\int_0^{\pi/2} x(2+\cos(x+y))\,dx\,dy$$

$$\int x\cos(x)\,dx = x\sin(x) + \cos(x)$$

$$= \int_0^{\pi/2} \left(\frac{\pi^2}{4} + \frac{\pi}{2}\sin\left(\frac{\pi}{2}+y\right) + \cos\left(\frac{\pi}{2}+y\right) - \cos(y) \right) dy$$

$$= \frac{\pi^3}{8} + \left[-\frac{\pi}{2} \cos\left(\frac{\pi}{2}+y\right) + \sin\left(\frac{\pi}{2}+y\right) - \sin(y) \right]_0^{\pi/2}$$

$$= \frac{\pi^3}{8} + \frac{\pi}{2} - 2$$

By symmetry, $M_y = M_x$. So

$$\bar{x} = \frac{M_y}{M} = \frac{\pi}{4} + \pi^{-1} - 4\pi^{-2}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\pi}{4} + \pi^{-1} - 4\pi^{-2}$$

Moments of inertia

The first moment told us about balance. The **second moment** tells us about kinetic energy in a rotating shaft.

Recall kinetic energy is

$$E_k = \frac{1}{2} m v^2$$

where m is mass and v is speed. Now consider an object of mass m rotating about an axis at constant angular velocity $\omega = \frac{d\theta}{dt}$:

If r is the distance to the axis of rotation, then the kinetic energy is

$$E_k = \frac{1}{2} m v^2 = \frac{1}{2} m \left(r \frac{d\theta}{dt} \right)^2$$

$$= \frac{1}{2} m r^2 \omega^2$$

If now D is a region possibly of varying density and varying distance to the axis (e.g. an object on a lathe), the kinetic energy stored will be

$$E_k = \iint_D \frac{1}{2} r^2 \delta(x,y)\, \omega^2 \, dA$$

r = distance to axis of rotation

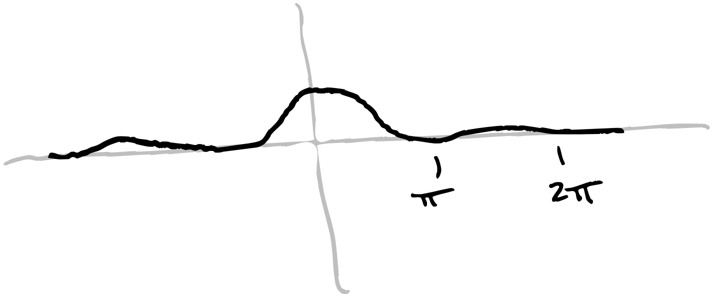
$$= \frac{1}{2} I \omega^2 \quad \text{where}$$

$$I = \iint_D r^2 \delta(x,y)\, dA$$

I is called the <u>second moment</u> or the <u>moment of inertia</u> of D relative to the axis.

<u>ex)</u> Find the moment of inertia with respect to the y-axis of a thin plate of constant density $\delta = 1\,g/cm^2$ bounded by the curve

$$y = \frac{\sin^2 x}{x^2} \quad \text{and} \quad \pi \le x \le 2\pi.$$



$$I = \int_{\pi}^{2\pi} \int_{0}^{\frac{\sin^2 x}{x^2}} x^2 \, dy \, dx$$

$$= \int_{\pi}^{2\pi} \sin^2 x \, dx$$

$$= \frac{x}{2} - \frac{\sin(2x)}{4} \Big]_{\pi}^{2\pi}$$

$$= \frac{\pi}{2}$$