§15.8 Substitutions in multiple integrals

Now we will discuss a general form of the chain rule. (Sort of already covered this in the lecture on differential forms.)

Suppose we want to integrate $f(x,y)\,dx\,dy$ over a region R.

$$\iint_R f(x,y)\,dx\,dy$$

Suppose x and y are themselves functions of independent variables u and v,

$$x = g(u,v), \qquad y = h(u,v).$$

The <u>preimage</u> G of R under (g,h) is

$$G = \{(u,v) \mid (x,y) = (g(u,v), h(u,v)) \in R\}$$

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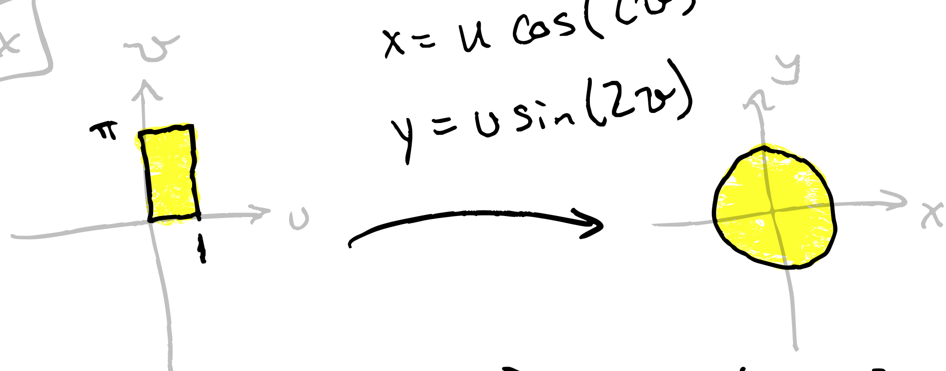
ex

$x = u\cos(2v)$

$y = u\sin(2v)$



$G = \{0 \le u \le 1,\ 0 \le v \le \pi\}$ $\qquad$ $R = \{x^2 + y^2 \le 1\}$

We will assume that (g, h) is one-to-one on the interior of G, and that it maps G onto R. That is,

- for every $(x, y) \in R$, there exists some (u, v) such that $g(u, v) = x$ and $h(u, v) = y$;

- if $g(u_1, v_1) = x = g(u_2, v_2)$ and
$$h(u_2, v_2) = y = h(u_2, v_2),$$
with (u_1, v_1) and (u_2, v_2) in the interior of G, then $(u_1, v_1) = (u_2, v_2)$.

The Jacobian determinant associated to this coordinate change is

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

We also denote this by $\frac{\partial(x,y)}{\partial(u,v)}$.

In diff. forms lecture I mistakenly wrote $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$. This will work out the same, but the above order is "better".

Then the higher-dimensional chain rule is:

<u>Thm</u> Suppose $f(x,y)$ is continuous on R. Let G be the preimage of R under $x = g(u,v)$, $y = h(u,v)$, with (g,h) one-to-one on the interior of G. If g and h have continuous first partial derivatives throughout G, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv$$

⚠ In the definition of $J(u,v)$ we used $|\;|$ for a determinant, but in the above formula we're taking the <u>absolute value</u> of $J(u,v)$.

This might look strange: after all, the one-variable chain rule is

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))\,g'(u)\,du$$

with no absolute value around $g'(u)$.

The reason is that the integral we learned in single-variable calculus is an <u>oriented</u> integral: we don't just integrate over $[a,b]$, like $\int_{[a,b]} f(x)\,dx$, we integrate <u>from</u> a <u>to</u> b, so $\int_a^b f(x)\,dx$.

And we <u>define</u> $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ for $a<b$.

We haven't discussed orientation

in higher dimensions (and we won't), but if we did, then the absolute value sign would go away. Similarly, in one-dimensional **unoriented** integration we do have

$$\int_{[a,b]} f(x)\,dx = \int_{g^{-1}[a,b]} f(g(u))\,|g'(u)|\,du.$$

Substitution for triple integrals is the same, where now

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Let's see some examples!

ex) Find

$$\int_1^3 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$

Let's make the substitution $u = \sqrt{\frac{y}{x}}$, $v = \sqrt{xy}$. We need to solve these for (x, y):

$$x = \frac{v}{u}, \quad y = uv.$$

The Jacobian is

$$J(u,v) = \begin{vmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ v & u \end{vmatrix}$$

$= -2\frac{v}{u}$. Since $\frac{v}{u} = x > 0$,

$|J(u,v)| = 2\frac{v}{u}$.

We also need to find the new bounds.

$1 \le y \le 3 \Rightarrow 1 \le uv \le 3$

$\frac{1}{y} \le x \le y \Rightarrow \frac{1}{uv} \le \frac{v}{u} \le uv$

The second set of inequalities tells us that $u \geq 1$ and $v \geq 1$.
Combining this with the first set of inequalities, our bounds

are

$$1 \leq v \leq 3$$

$$1 \leq u \leq 3/v$$

So $$\int_1^3 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$

$$= \int_1^3 \int_1^{3/v} u e^{v} \left(\frac{2v}{u}\right) du\, dv$$

$$= 2\int_1^3 \int_1^{3/v} v e^{v} \, du \, dv$$

$$= 2\int_1^3 (3e^{v} - v e^{v}) \, dv$$

Integration by parts $\Rightarrow$

$\int v e^{v} dv = v e^{v} - e^{v}$, so

$$= 2\left[3e^{3} - 3e^{3} + e^{3} - (3e - e + e)\right] = 2e(e^{2} - 3)$$

ex] Find

$$\int_0^{\pi}\int_0^{2y} \cos(x-2y)(x+y)\,dx\,dy$$

We'll substitute $u = x - 2y$

$v = x + y$

① Solve for x and y:

$$x = \frac{u+2v}{3}$$

$$y = \frac{-u+v}{3}$$

② Find Jacobian determinant!

$$J(u,v) = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -1/3 & 1/3 \end{vmatrix} = \frac{1}{3}$$

③ Find new bounds:

$$0 \le y \le \pi \Rightarrow 0 \le -u + v \le 3\pi$$

$$\Rightarrow u \le v \le u + 3\pi$$

$$0 \le x \le 2y \Rightarrow -2y \le u \le 0$$

$$-2y \le u \Rightarrow 2u - 2v \le 3u$$

$$\Rightarrow -\frac{u}{2} \le v$$

Our bounds for u are thus $-2\pi \leq u \leq 0$. For v, we have

$$u \leq v \leq u + 3\pi$$

<u>but also</u> $-u/2 \leq v$. Since $u \leq 0$, $u \leq -\frac{u}{2}$, and thus our v bounds are $-\frac{u}{2} \leq v \leq u + 3\pi$.

(4) Evaluate the integral:

$$\int_0^{\pi}\int_0^{2y} \cos(x-2y)(x+y)\,dx\,dy$$

$$= \frac{1}{3}\int_{-2\pi}^{0}\int_{-u/2}^{u+3\pi} \cos(u)\,v \; dv\,du$$

$$= \frac{1}{3} \int_{-2\pi}^{0} \cos(u) \, \frac{u^2 + 6\pi u + 9\pi^2 - u^2/4}{2} \, du$$

$$= \int_{-2\pi}^{0} \cos(u) \left(\frac{u^2}{8} + \pi u \right) du + \cancelto{0}{\frac{3\pi^2}{2} \int_{-2\pi}^{0} \cos(u) \, du}$$

$$= \cancelto{0}{\left(\frac{u^2}{8} + \pi u \right) \sin(u) \Big]_{-2\pi}^{0}} - \int_{-2\pi}^{0} \frac{u}{4} \sin(u) \, du$$

$$= \frac{1}{4} \Big[u \cos(u) - \sin(u) \Big]_{-2\pi}^{0}$$

$$= \frac{\pi}{2}.$$