§16.1 Line integrals of scalar functions

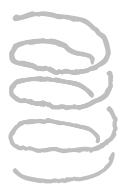
Want to integrate a function over a <u>curve</u> instead of just intervals $[a,b]$.

<u>e.g.</u> mass of a wire, work done along a curve, ...

So let $f(x,y,z)$ be a function,

and C a curve with parametrization

$$\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k},$$
$$a \le t \le b.$$



$$\vec{r}(t) = 2\cos(t)\,\vec{i} + 2\sin(t)\,\vec{j} + t\,\vec{k}$$

We divide C into n subarcs, and pick one point (x_k, y_k, z_k) from each. The associated Riemann sum is

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k$$

where Δs_k is the length of the small subarc. Note we are taking the length of the curve itself, <u>not</u> the length in the 't' variable! We want our definition to depend only on the curve C, <u>not</u> a particular parametrization of it.

As usual, we then take the limit as $n \to \infty$ with Δs_k going to 0.

<u>Def</u> The <u>line integral</u> of f over C is

$$\int_C f(x,y,z)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k$$

(provided this limit exists).

If C is smooth, meaning

$\vec{v}(t) = \vec{r}\,'(t)$ is continuous

and never $\vec{0}$, and f is

continuous on C, then the

limit exists. These assumptions

will be satisfied in all the

cases we're interested in.

So how do we evaluate this

integral? We have seen that

the arclength parametrization

of a curve is generally

difficult or impossible to find. Fortunately, we don't need to!

Let $\vec{r}(t)$, $a \le t \le b$ be a smooth parametrization of C.

Recall the arclength $s(t)$ of C is

$$s(t) = \int_a^t |\vec{r}\,'(\tau)|\, d\tau$$

and thus $ds = \frac{ds}{dt}\, dt = |\vec{r}\,'(t)|\, dt.$

This gives

$$\int_C f(x,y,z)\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

$\vec{r}'(t)$ is also often written $\vec{v}(t)$.

ex) Evaluate $\int_C \sqrt{x^2+y^2}\, ds$ over

the helix C parametrized by

$\vec{r}(t) = 2\cos(t)\,\vec{i} + 2\sin(t)\,\vec{j} + t\,\vec{k}$,
$0 \le t \le 4\pi$

① Find $\vec{r}'(t)$:

$\vec{r}'(t) = -2\sin(t)\,\vec{i} + 2\cos(t)\,\vec{j} + \vec{k}$

② Find $|\vec{r}'(t)|$:

$|\vec{r}'(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1^2}$
$= \sqrt{5}$

③ Integrate: note $\sqrt{x^2+y^2} = 2$

on C, so

$$\int_C \sqrt{x^2+y^2}\, ds = \int_0^{4\pi} 2\sqrt{5}\, ds = 8\pi\sqrt{5}$$

ex) Evaluate $\int_C \sqrt{x^2+y^2}\, ds$ when

C is the straight line segment from

$(2,0,0)$ to $(2,0,4\pi)$.

Let $\vec{r}(t) = (2, 0, 4\pi t)$ for

$0 \le t \le 1$. Then

$$\vec{r}\,'(t) = 4\pi \vec{k}$$

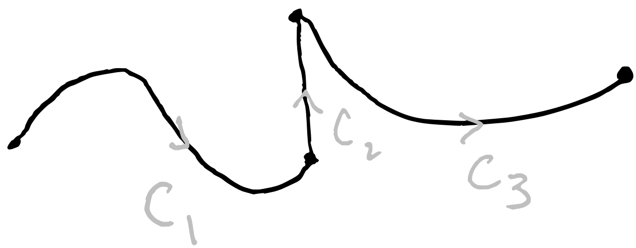
and so $|\vec{r}\,'(t)| = 4\pi$. Thus

$$\int_C \sqrt{x^2+y^2}\, ds = \int_0^1 (2)(4\pi)\, dt = 8\pi$$

In both of these examples, the curve began at $(2,0,0)$ and ended at $(2,0,4\pi)$. Thus:

$\boxed{\text{The value of a line integral along a path joining two points } \vec{a} \text{ and } \vec{b} \text{ can change if you use a different path.}}$

Rmk If C is made by joining curves $C_1, \ldots, C_n$ end to end,



as pictured here, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \cdots + \int_{C_n} f\,ds.$$

So what are these good for?

Interpretations of line integrals

- Suppose C represents a wire (or spring, rod, etc) in space. If $\delta(x,y,z)$ is the density (per unit length), then the mass of the wire is

$$M = \int_C \delta \, ds$$

We have analogous formulas for the moments:

First moments about the coordinate planes:

$$M_{yz} = \int_C x\,\delta\,ds, \quad M_{xz} = \int_C y\,\delta\,ds,$$

$$M_{xy} = \int_C z\,\delta\,ds$$

Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \quad \bar{y} = M_{xz}/M,$$

$$\bar{z} = M_{xy}/M$$

Moments of inertia about coordinate axes:

$$I_x = \int_C (y^2 + z^2)\,\delta\, ds, \quad I_y = \int_C (x^2 + z^2)\,\delta\, ds,$$

$$I_z = \int_C (x^2 + y^2)\,\delta\, ds$$

- if C is a plane curve,

$$\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j},$$

and $f(x,y)$ is a real-valued function, then $\int_C f\, ds$ is the area of the "wall" $\{(g(t), h(t), \tau f(g(t), h(t)) \mid a \le t \le b,\ 0 \le \tau \le 1\}$

ex] Find the center of mass of a wire lying along the curve

$\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + (1-t)\,\vec{k}, \quad 0 \le t \le 1$

with density $\delta(t) = \sqrt{2t^2+1}$.

$$\vec{r}'(t) = \vec{i} + 2t\vec{j} - \vec{k}$$

$$\Rightarrow |\vec{r}'(t)| = \sqrt{1+4t^2+1} = \sqrt{2}\sqrt{2t^2+1}$$

Mass:

$$M = \int_C \delta \, ds = \int_0^1 \left(\sqrt{2t^2+1}\right)\left(\sqrt{2}\sqrt{2t^2+1}\right) dt$$

$$= \sqrt{2}\int_0^1 (2t^2+1)\, dt = \frac{5\sqrt{2}}{3}$$

Moments:

$$M_{yz} = \int_C x\,\delta\, ds = \int_0^1 t\sqrt{2}(2t^2+1)\,dt$$

$$= \sqrt{2}\int_0^1 (2t^3 + t)\,dt = \sqrt{2}$$

$$M_{xz} = \int_C y\,\delta\,ds = \int_0^1 (t^2)\sqrt{2}(2t^2+1)\,dt$$

$$= \sqrt{2}\int_0^1 (2t^4 + t^2)\,dt = \frac{11\sqrt{2}}{15}$$

$$M_{xy} = \int_C z\,\delta\,ds = \int_0^1 (1-t)\sqrt{2}(2t^2+1)\,dt$$

$$= \sqrt{2}\int_0^1 (-2t^3 + 2t^2 - t + 1)\,dt$$

$$= \frac{2\sqrt{2}}{3}$$

So center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right)$$

$$= \left(\frac{\sqrt{2}}{5\sqrt{2}/3}, \frac{11\sqrt{2}/15}{5\sqrt{2}/3}, \frac{2\sqrt{2}/3}{5\sqrt{2}/3} \right)$$

$$= \left(\frac{3}{5}, \frac{11}{25}, \frac{2}{5} \right).$$