§16.2 Vector fields and line integrals

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Vector fields

A vector field is a function assigning a vector to each point in space (or to each point in its domain). It has the form

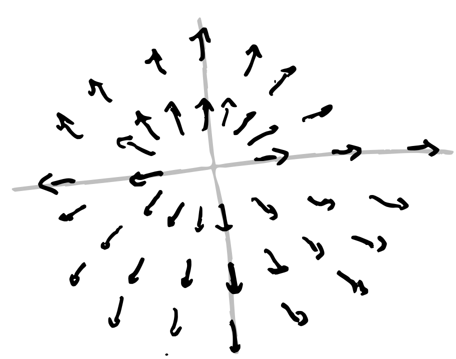
$$\vec{F} = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$$

We depict a vector field by sampling several points and drawing $\vec{F}(x_k, y_k, z_k)$ starting at (x_k, y_k, z_k).

ex The radial vector field is

$$\frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

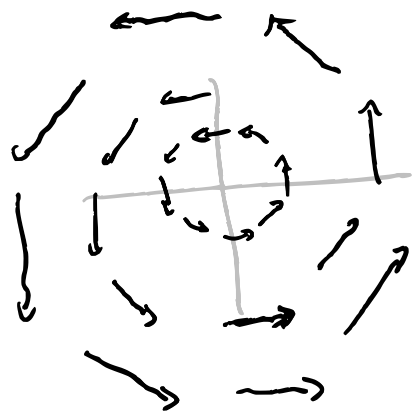
(this is not defined at the origin). In two dimensions, this looks like



<u>ex</u> The <u>angular vector field</u>

on $\mathbb{R}^2 - \{(0,0)\}$ is

$$-y\vec{i} + x\vec{j}$$



<u>Rmk</u> As I said in the differential forms lecture, the constant vector field $\vec{F}(x,y,z) = \vec{i}$ is sometimes denoted by $\frac{\partial}{\partial x}$, and similarly $\frac{\partial}{\partial y}$ for $\vec{F}(x,y,z) = \vec{j}$, $\frac{\partial}{\partial z}$ for $\vec{k}$.

In this notation, the radial vector field is

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}$$

$$= \frac{x}{r}\vec{\imath} + \frac{y}{r}\vec{\jmath}$$

while the angular vector field is

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}$$

$$= -y\,\vec{\imath} + x\,\vec{\jmath}.$$

Some "real-world" examples of vector fields are:

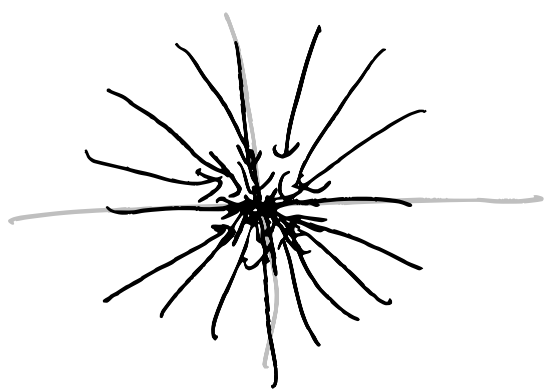
- the rate + direction a fluid (wind, water, etc.) is flowing in a domain
- a gravitational/magnetic/etc. force field

<u>Note</u> In physics and engineering, vector fields are often just called "vectors".

If $f(x,y,z)$ is a real-valued function, we can view the gradient of f as a vector field:

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

For example, suppose $T(x,y,z)$ describes the temperature at the point (x,y,z). Recall the gradient at each point points in the direction where T is increasing the fastest. Thus, the vector field ∇T will "point toward the heat source".



$$T = 20 - \frac{1}{2}(x^2+y^2)$$

$$\nabla T = -x\vec{i} - y\vec{j}$$

Line Integrals of Vector Fields

Vector fields can be integrated over curves. Let $\vec{F}$ be a vector field, and let C be a curve parametrized by $\vec{r}(t)$, $a \le t \le b$.

Recall that the unit tangent vector $\vec{T}$ along C is equivalently the tangent vector $\vec{r}\,'(t)$ divided by its length, or the tangent vector for the arclength parametrization:

$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{d\vec{r}}{ds}$$

(As usual we assume that C is **smooth** meaning $\vec{r}\,'(t) \neq 0$ for all t).

Our integral for vector fields will extract, at each point (x, y, z) along C, the component of $\vec{F}$ along $\vec{r}\,'(t)$.

Def The integral of $\vec{F}$ over C is

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot \frac{d\vec{r}}{ds}\, ds$$

$$= \int_C F(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_C \vec{F} \cdot d\vec{r}$$

For example, if $\vec{F}$ describes a force field, then this is the work done by $\vec{F}$ moving an object along C.

$$W = \int_C \vec{F} \cdot \vec{T}\, ds$$

ex) Find the work done by

$$\vec{F} = -y\,\vec{i} + x\,\vec{j} - z\,\vec{k}$$

along

(a) the straight line segments
$(0,0,0) - (1,1,0) - (1,1,1)$;

(b) the curve $\vec{r}(t) = (t, t^2, t^3)$, $0 \le t \le 1$.

a) Let C_1 be the line segment from $(0,0,0)$ to $(1,1,0)$, and C_2 the segment from $(1,1,0)$ to $(1,1,1)$. These are parametrized by

$$C_1: \ (t,t,0) \qquad 0 \le t \le 1$$
$$C_2: \ (1,1,t) \qquad 0 \le t \le 1$$

so

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_0^1 (-t\vec{i} + t\vec{j}) \cdot (\vec{i} + \vec{j})\, dt + \int_0^1 (-\vec{i} + \vec{j} - t\vec{k}) \cdot (\vec{k})\, dt$$

$$= \int_0^1 -t \, dt = -\frac{1}{2}$$

b) $$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left(-t^2 \vec{i} + t \vec{j} - t^3 \vec{k}\right) \cdot \left(\vec{i} + 2t \vec{j} + 3t^2 \vec{k}\right) dt$$

$$= \int_0^1 \left(-t^2 + 2t^2 - 3t^5\right) dt$$

$$= -\frac{1}{6}.$$

If $\vec{F}$ is the velocity field of a fluid, the integral is called the <u>flow</u> of $\vec{F}$ along C. If C is a <u>loop</u>, meaning it starts and ends at the same point, then we also call it the <u>circulation</u> of $\vec{F}$ along C.

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T} \, ds$$

$\underline{ex}$ Find the circulation of

a) $\vec{F} = -y\,\vec{i} + x\,\vec{j} + \vec{k}$

b) $\vec{F} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

around the loop

$\vec{r}(t) = \cos t\,\vec{i} + \sin(t)\,\vec{j}\,, \quad 0 \le t \le 2\pi.$

a) Circulation $= \int_C \vec{F} \cdot d\vec{r}$

$$= \int_0^{2\pi} \left(-\sin t\,\vec{i} + \cos t\,\vec{j}\right) \cdot \left(-\sin t\,\vec{i} + \cos(t)\,\vec{j}\right) dt$$

$$= 2\pi$$

b) Circulation $= \int_C \vec{F} \cdot d\vec{r}$

$$= \int_0^{2\pi} \left(\cos(t)\,\vec{i} + \sin(t)\,\vec{j} \right) \cdot \left(-\sin t\,\vec{i} + \cos t\,\vec{j} \right) dt$$

$= 0.$

__Line integrals with respect to dx, dy, dz__

Just as it is often useful to decompose a force into components, we often want to decompose a line integral into components.

If $f(x,y,z)$ is a real-valued function defined on a curve C, we define

$$\int_C f\,dx = \int_C \left(f(x,y,z)\,\vec{i}\right) \cdot d\vec{r}$$

$$\int_C f\,dy = \int_C \left(f(x,y,z)\,\vec{j}\right) \cdot d\vec{r}$$

$$\int_C f\,dz = \int_C \left(f(x,y,z)\,\vec{k}\right) \cdot d\vec{r}$$

With this notation, if

$$\vec{F} = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k},$$

then

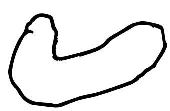
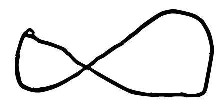
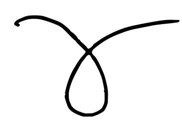
$$\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + \int_C N\,dy + \int_C P\,dz.$$

$$= \int_C M\,dx + N\,dy + P\,dz$$

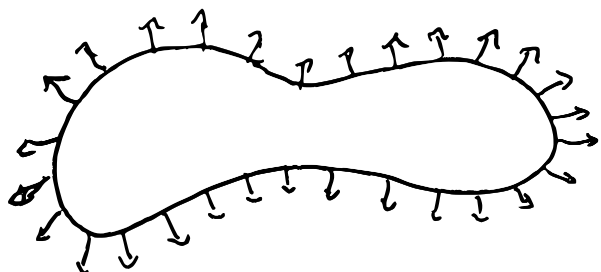
Flux

We can also use line integrals to calculate the amount of fluid flowing out of a region. First, some terminology:

A curve is <u>simple</u> if it does not cross itself. A <u>closed curve</u> is another name for a loop.

	Simple	Not simple
Closed		
Not closed		

If C is a simple closed curve parametrized by $\vec{r}(t)$, $a \le t \le b$, the <u>flux</u> of $\vec{F}$ over (the region enclosed by) C extracts the component of $\vec{F}$, not in the <u>tangent</u> direction $\vec{T}$ of C, but the <u>outward-pointing unit normal</u> $\vec{n}$ of C:



$$\boxed{\text{Flux} = \int_C \vec{F} \cdot \hat{n} \, ds}$$

How do we find $\vec{n}$ from $\vec{T}$?

We can get a vector orthogonal to $\vec{T}$ by crossing with $\vec{k}$:

$$\vec{n} = \vec{T} \times \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j}$$

$\boxed{!}$ In order for this to point <u>outward</u>, this assumes C is parametrized such that C is traversed <u>counterclockwise</u> as t increases. If C is traversed <u>clockwise</u>, we should instead take

$$\vec{n} = \vec{k} \times \vec{T} = -\frac{dy}{ds}\vec{i} + \frac{dx}{ds}\vec{j}$$

If $\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$,

then

$$\vec{F}\cdot\vec{n} = M(x,y)\frac{dy}{ds} - N(x,y)\frac{dx}{ds}$$

so

$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)ds$$

$$= \oint_C M\,dy - N\,dx$$

Here, the $\oint_C$ means the same

thing as $\int_C$, we just add the circle to emphasize that C is a <u>closed</u> curve. Sometimes you will also see $\oint_C$ to emphasize that C is being traversed counterclockwise.

<u>ex</u> Find the flux of

$$\vec{F} = (x-y)\vec{i} + (x+y)\vec{j}$$

across the circle $x^2+y^2=1$.

We parametrize the circle C via

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}.$$

Then $dy = d(\sin t) = \cos t \, dt$

$dx = d(\cos t) = -\sin t \, dt$

so

$$\text{Flux} = \oint_C (x-y)\,dy - (x+y)\,dx$$

$$= \int_0^{2\pi} \left[(\cos t - \sin t)\cos t + (\cos t + \sin t)\sin t\right] dt$$

$$= \int_0^{2\pi} dt = 2\pi$$