§16.3 Path independence, conservative fields, and potential functions

We have seen that, in general, the integral of a vector field $\vec{F}$ along a curve C from a point A to B depends on the curve C, not just on the endpoints A and B. Perhaps surprisingly, there are many vector fields for which this is <u>not</u> the case.

<u>Defn</u> Let $\vec{F}$ be a vector field defined on an open region D in space. We say that $\vec{F}$ is <u>conservative</u> if, for every pair of points $A, B \in D$, we have

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

for any two curves C_1, C_2 connecting A to B and lying in D.

When $\vec{F}$ is conservative, we may write $\int_A^B \vec{F}\cdot d\vec{r}$ instead of $\int_C \vec{F}\cdot d\vec{r}$ to emphasize this "path-independence".

<u>Rmk</u> The term "conservative" comes from physics, since it is related to conservation of energy. For example, <u>gravitational fields</u> $\vec{G}$ and <u>electric fields</u> $\vec{E}$

are conservative. A gravitational field describes the effect of gravity; the force exerted on an object of mass m is $\vec{F} = m\vec{G}$. Similarly, the force exerted by an electric field on an object of charge q is $\vec{F} = q\vec{E}$.

So, the fact that these types of fields are conservative translates to: in a gravitational/electric field, the work needed to move a mass/charge from point A to point B is the same no matter how we get from A to B.

Under mild conditions, we will show that a vector field $\vec{F}$ is conservative if and only if it is "antidifferentiable", i.e. if and only if there is a function f with $\nabla f = \vec{F}$. We call f a <u>potential function</u>

for $\vec{F}$ (another term from physics). In this case,

$$\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A),$$

a vector version of the fundamental version of calculus.

<u>ex</u> Let $f(x,y,z) = -\dfrac{1}{x^2+y^2+z^2}$.

Find the work done by $\vec{F} = \nabla f$ in moving from

$(1,0,0)$ to $(-2,0,0)$ via any path not passing through the origin.

$$\int_{(1,0,0)}^{(-2,0,0)} \vec{F}\cdot d\vec{r} = -\frac{1}{x^2+y^2+z^2}\Bigg]_{(1,0,0)}^{(-2,0,0)}$$

$$= \frac{3}{4}.$$

For illustration, let's calculate this directly. The straight line segment from $(1,0,0)$ to $(-2,0,0)$ passes through the origin, so we need a different path. Let's take the "spiral"

$$\vec{r}(t) = (1+t)\cos(\pi t)\vec{i} + (1+t)\sin(\pi t)\vec{j}$$

We have

$$\vec{F} = \frac{2}{(x^2+y^2+z^2)^2}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$\vec{F}(r(t)) = \frac{2}{(1+t)^3}\left(\cos(\pi t)\vec{i} + \sin(\pi t)\vec{j}\right)$$

$$d\vec{r} = \Big(\cos(\pi t) - \pi(1+t)\sin(\pi t)\Big)\vec{i} + \Big(\sin(\pi t) + \pi(1+t)\cos(\pi t)\Big)\vec{j}$$

$$dt$$

so $\vec{F} \cdot d\vec{r} = \dfrac{2}{(1+t)^3}\,dt.$

Thus

$$\int_{(1,0,0)}^{(-2,0,0)} \vec{F} \cdot d\vec{r} = \int_0^1 \frac{2}{(1+t)^3}\,dt$$

$$= \left[-\frac{1}{(1+t)^2} \right]_0^1 = \frac{3}{4}.$$

Another option would have been to take the semicircle from $(1,0,0)$ to $(-1,0,0)$ followed by the straight line segment from $(-1,0,0)$ to $(-2,0,0)$.

Here are our main theorems describing conservativity.

"if and only if"

Thm 1 $\vec{F}$ is conservative $\iff$ $\vec{F} = \nabla f$, in which case

$$\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

Thm 2 $\vec{F}$ is conservative $\iff$

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

for every loop C in the domain D of $\vec{F}$.

Thm 2 is easier to prove, so let's start with that one.

<u>Pf of Thm 2</u> "$\Rightarrow$" Let C be a loop parametrized by $\vec{r}(t)$, $0 \le t \le 1$, so that $P = \vec{r}(0) = \vec{r}(1)$. By assumption,

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r}$$

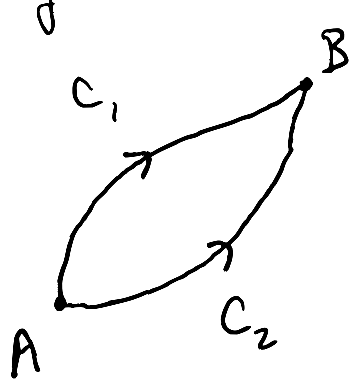
for any other loop C_2 based at P.

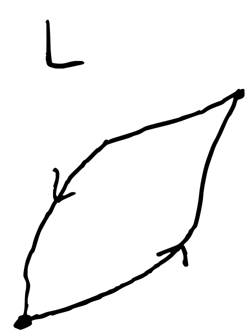
If we take C_2 to be the

constant loop $\vec{r}_2(t) = P$, then

$$\oint_{C_2} \vec{F} \cdot d\vec{r} = 0.$$

"$\Leftarrow$" Let C_1, C_2 be two paths from A to B. If we reverse C_1 and concatenate it with C_2, we get a loop L starting and ending at A.





Now $\displaystyle\int_{C_2} \vec{F}\cdot d\vec{r} - \int_{C_1} \vec{F}\cdot d\vec{r}$

$$= \oint_{L} \vec{F}\cdot d\vec{r} = 0$$

so $\displaystyle\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}.$ $\square$

<u>Pf of Thm 1</u> "$\Leftarrow$" Let

$$\vec{F} = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

and let C be a curve in D parametrized by $\vec{r}(t)$, $0 \le t \le 1$, with $\vec{r}(0) = A$ and $\vec{r}(1) = B$.

If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{i} + z(t)\vec{k}$,

then

$$\vec{F} \cdot d\vec{r} = \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) \cdot \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right) dt$$

$$= \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt$$

$$= \frac{df}{dt}\,dt = d(f \circ \vec{r})$$

So $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 d(f \circ \vec{r})$

$= f(\vec{r}(1)) - f(\vec{r}(0))$

$= f(B) - f(A).$

Note that once we have this result, we can easily see the "$\Rightarrow$" direction of Thm 2.

$$\oint_L \vec{F} \cdot d\vec{r} = f(A) - f(A) = 0.$$

The most difficult part is the "$\Rightarrow$" part of Thm 1.

"$\Rightarrow$" Choose a point A in the domain D of $\vec{F}$; we assume that D is open and connected.

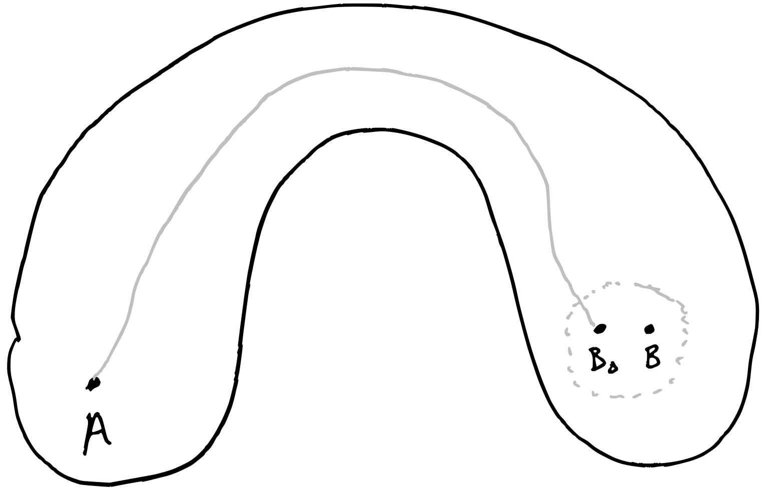
For any B in D, define

$$f(B) = \int_A^B \vec{F} \cdot d\vec{r}$$

By assumption, this is well-defined.

We need to show that $\nabla f = \vec{F}$, or in other words that $\frac{\partial f}{\partial x} = \vec{F} \cdot \vec{i}$,

$$\frac{\partial f}{\partial y} = \vec{F} \cdot \vec{j}, \quad \frac{\partial f}{\partial z} = \vec{F} \cdot \vec{k}.$$



Let $B = (x_1, y_1, z_1)$. Since D is open, there is some small ball centered at B which is contained in D.

In particular, there is a point $B_0 = (x_0, y_1, z_1)$ such that the entire line segment

$$L: (x_0 + t(x_1 - x_0), y_1, z_1)$$

from B_0 to B is contained in D. Now

$$f(x_0+t, y_1, z_1) = \int_A^{(x_0+t, y_1, z_1)} \vec{F} \cdot d\vec{r}$$

$$= \int_A^{B_0} \vec{F} \cdot d\vec{r} + \int_{B_0}^{(x_0+t, y_1, z_1)} \vec{F} \cdot d\vec{r}$$

$$= \underbrace{\int_A^{B_0} \vec{F} \cdot d\vec{r}}_{\text{constant}} + \int_0^t \vec{F}(x_0+\tau, y_1, z_1) \cdot \vec{i} \, d\tau$$

By the FTC from single-variable calculus, we have

$$\frac{\partial f}{\partial x} = \frac{d}{dt}\int_0^t \vec{F}\cdot\vec{i}\,d\tau = \vec{F}\cdot\vec{i}.$$

A similar argument shows that

$\frac{\partial f}{\partial y} = \vec{F}\cdot\vec{j}$ and $\frac{\partial f}{\partial z} = \vec{F}\cdot\vec{k}$, so we are done. $\square$

Having $\oint_L \vec{F}\cdot d\vec{r} = 0$ for all loops L is a useful thing to know about conservative vector fields, and if we can find a loop L such that

$\oint_L \vec{F}\cdot d\vec{r} \neq 0$, then we can conclude that $\vec{F}$ is not conservative.

However, it isn't feasible to directly calculate $\oint_L \vec{F} \cdot d\vec{r}$ for <u>every</u> loop L, so we still don't have an easy way to show that a vector field <u>is</u> conservative.

We can do better with the gradient characterization. Suppose

$$\vec{F} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}.$$

If $\vec{F} = \nabla f$, so that $\frac{\partial f}{\partial x} = M$, $\frac{\partial f}{\partial y} = N$, $\frac{\partial f}{\partial z} = P$, then by symmetry of mixed partials we must have

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial N}{\partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial P}{\partial y}$$

$$\frac{\partial P}{\partial x} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial M}{\partial x}$$

<u>Rmk</u> Abusing notation slightly, write

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.$$

The "cross product"

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

is called the <u>curl</u> of $\vec{F}$; we will study it more in the coming sections. The preceding equations show that a necessary condition for $\vec{F}$ to be conservative is $\nabla \times \vec{F} = 0$.

ex| Let

$$\vec{F} = xy\,\vec{i} + xz\,\vec{j} + yz\,\vec{k}.$$

Since $\frac{\partial}{\partial y}(xy) = x \neq z = \frac{\partial}{\partial x}(xz)$,

$\vec{F}$ is not conservative.

So $\nabla \times \vec{F} = 0$ is a <u>necessary</u> condition for $\vec{F}$ to be conservative.

Whether it is <u>sufficient</u> depends on the shape of the domain D, as the following example shows.

$\downarrow$

ex) Let

$$\vec{F} = \frac{1}{x^2+y^2}\left(-y\,\vec{i} + x\,\vec{j}\right)$$

Then $\nabla \times \vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= 0$$

So $\vec{F}$ satisfies the component test.

However, $\vec{F}$ is not conservative if C is the unit circle, with standard parametrization

C: $\cos(t)\vec{i} + \sin(t)\vec{j} \quad 0 \le t \le 2\pi$

then $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(-\sin(t)\vec{i} + \cos(t)\vec{j}\right) \cdot \left(-\sin(t)\vec{i} + \cos(t)\vec{j}\right) dt$

$\displaystyle = \int_0^{2\pi} dt = 2\pi.$

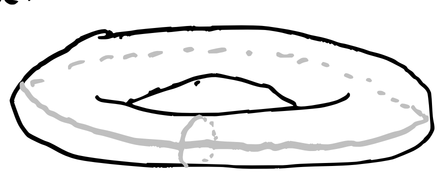
This is related to the fact that

$\vec{F} = \frac{1}{x^2+y^2}\left(-y\vec{i} + x\vec{j}\right)$ is not

defined at the origin. In some sense, this example turns out to be the only thing that can go wrong:

Def D is simply connected if every loop in D can be contracted to a point.

For example, $\mathbb{R}^n$ is simply connected. $\mathbb{R}^2 - \{(0,0)\}$ is not simply connected: a circle enclosing the "missing" origin cannot be contracted to a point. A sphere is simply connected, but a torus is not.



Rmk Most sources require simply connected regions to also be connected, but the textbook doesn't.

Now:

Thm Let $\vec{F}$ be a vector field on an open, simply connected domain D, whose component functions have continuous first partial derivatives. If $\nabla \times \vec{F} = 0$, then $\vec{F}$ is conservative.

We will discuss this further in §16.7.

ex) Show that

$$\vec{F} = y e^{xy} \vec{i} + \left(x e^{xy} + \cos(z) \right) \vec{j} + \left(1 - y \sin(z) \right) \vec{k}$$

is conservative, and find a potential function for it.

We apply the component test!

$$\frac{\partial}{\partial y}\left(y e^{xy} \right) = e^{xy} + xy e^{xy}$$

$$= \frac{\partial}{\partial x}\left(x e^{xy} + \cos(z) \right)$$

$$\frac{\partial}{\partial z}\left(y e^{xy} \right) = 0 = \frac{\partial}{\partial x}\left(1 - y \sin(z) \right)$$

$$\frac{\partial}{\partial z}\left(x e^{xy} + \cos(z) \right) = -\sin(z) = \frac{\partial}{\partial y}\left(1 - y \sin(z) \right)$$

Since $\vec{F}$ is defined on all of $\mathbb{R}^3$, which is simply connected, $\vec{F}$ must be conservative. We are trying to solve the equations

$$\frac{\partial f}{\partial x} = y e^{xy}$$

$$\frac{\partial f}{\partial y} = x e^{xy} + \cos(z)$$

$$\frac{\partial f}{\partial z} = 1 - y \sin(z)$$

We start by integrating the

first equation, treating y and z as constants'.

$$f = \int y e^{xy} \, dx = e^{xy} + g(y,z)$$

where $g(y,z)$ is a constant in terms of x, but depends on y and z.

Now we apply $\frac{\partial}{\partial y}$ to get

$$x e^{xy} + \cos(z) = \frac{\partial f}{\partial y}$$

$$= x e^{xy} + \frac{\partial g}{\partial y}$$

so $\frac{\partial g}{\partial y} = \cos(z)$.

We integrate this with respect to y, treating z as a constant, to get

$$g = \int \cos(z)\, dy = y\cos(z) + h(z)$$

So $f = e^{xy} + y\cos(z) + h(z)$.

Finally,

$$1 - y\sin(z) = \frac{\partial f}{\partial z} = -y\sin(z) + \frac{dh}{dz}$$

$$\Rightarrow \frac{dh}{dz} = 1 \Rightarrow h = z + C$$

so $f(x,y,z) = e^{xy} + y\cos(z) + z + C$

ex) Show that

$$\vec{F} = 2xz\,\vec{i} - z\sin(yz)\,\vec{j} + (x^2 - y\sin(yz))\,\vec{k}$$

is conservative, and find a potential function for it.

We apply the Component Test!

$$\frac{\partial}{\partial y}(2xz) = 0 = \frac{\partial}{\partial x}(-z\sin(yz))$$

$$\frac{\partial}{\partial z}(2xz) = 2x = \frac{\partial}{\partial x}(x^2 - y\sin(yz))$$

$$\frac{\partial}{\partial z}(-z\sin(yz)) = -\sin(yz) - yz\cos(yz)$$

$$= \frac{\partial}{\partial y}(2x - y\sin(yz))$$

Since $\vec{F}$ is defined on all of $\mathbb{R}^3$, which is simply connected, it must be conservative. So:

$$f = \int 2xz\,dx = x^2 z + g(y,z)$$

$$-z\sin(yz) = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$$

$$\Rightarrow g = \int (-z\sin(yz))\, dy$$

$$= \cos(yz) + h(z)$$

So $f = x^2 z + \cos(yz) + h(z)$

Now $\frac{\partial}{\partial z}(x^2 z + \cos(yz) + h(z))$

$$= 2xz - y\sin(yz) + \frac{\partial h}{\partial z}$$

$$= \frac{\partial f}{\partial z} + \frac{\partial h}{\partial z}$$

So $\frac{\partial h}{\partial z} = 0$, i.e. $h = C$. So

$f = x^2 z + \cos(yz) + C$

Exact differential forms

Consider a line integral

$$\int_C M\,dx + N\,dy + P\,dz$$

Recall that the differential of a real-valued function f is

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

If $M\,dx + N\,dy + P\,dz$ is of this form, then the line integral is easy to evaluate!

$$\int_C \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

∇f is conservative

$$= \int_C \nabla f \cdot d\vec{r}$$

$$= \int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$$

<u>Def</u> A <u>differential 1-form</u> is an expression

$$M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$$

A 1-form is <u>exact</u> if

$M\,dx + N\,dy + P\,dz = df$.

This is equivalent to the vector field $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ being conservative.

There is a version of the Component Test:

<u>Def</u> A 1-form $M\,dx + N\,dy + P\,dz$ is <u>closed</u> if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

From symmetry of mixed partials, exact $\Rightarrow$ closed. Conversely!

<u>Thm</u> On an open, <u>simply connected</u> domain D, every closed 1-form is exact.

<u>Rmk</u> The differential of a 1-form

$$\omega = M\,dx + N\,dy + P\,dz$$

is defined by

$$d\omega = \cancel{\frac{\partial M}{\partial x} dx\,dx} + \frac{\partial M}{\partial y} dy\,dx + \frac{\partial M}{\partial z} dz\,dx$$

$$+ \frac{\partial N}{\partial x} dx\,dy + \cancel{\frac{\partial N}{\partial y} dy\,dy} + \frac{\partial N}{\partial z} dz\,dy$$

$$+ \frac{\partial P}{\partial x} dx\,dz + \frac{\partial P}{\partial y} dy\,dz + \cancel{\frac{\partial P}{\partial z} dz\,dz}$$

Using the convention

$dy\,dx = -dx\,dy$, and thus

$dx\,dx = 0$, $dy\,dy = 0$, $dz\,dz = 0$,

this becomes

$$d\omega = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$+ \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) dx\,dz$$

$$+ \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) dy\,dz$$

So ω is closed $\Longleftrightarrow d\omega = 0$, and the statement "exact $\Rightarrow$ closed" is just $d(df) = 0$.

An expression

$$M(x,y,z)\,dx\,dy + N(x,y,z)\,dx\,dz + P(x,y,z)\,dy\,dz$$

is called a <u>differential 2-form</u>.

<u>ex</u> Show that

$$\omega = 2xyz\,dx + x^2 z\,dy + x^2 y\,dz$$

is exact, and evaluate

$$\int_{(2,-1,1)}^{(3,1,1)} 2xyz\,dx + x^2 z\,dy + x^2 y\,dz$$

We use the component test:

$$\frac{\partial}{\partial y}(2xyz) = 2xz = \frac{\partial}{\partial x}(x^2 z)$$

$$\frac{\partial}{\partial z}(2xyz) = 2xy = \frac{\partial}{\partial x}(x^2 y)$$

$$\frac{\partial}{\partial z}(x^2 z) = x^2 = \frac{\partial}{\partial y}(x^2 y)$$

So ω is closed, hence exact since $\mathbb{R}^3$ is simply connected. We find f such that $\omega = df$ by integrating:

$$f = \int 2xyz\, dx = x^2 yz + g(y,z)$$

We see that $f = x^2yz$ already works, so $g = C$. Now

$$\int_{(2,-1,1)}^{(3,1,1)} \omega = \int_{(2,-1,1)}^{(3,1,1)} df = x^2yz \Big]_{(2,-1,1)}^{(3,1,1)}$$

$$= 13$$