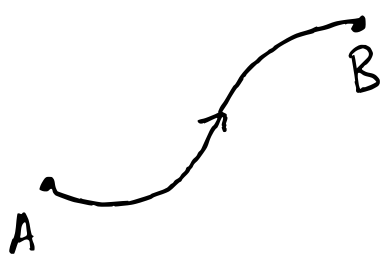
§16.4 Green's theorem in the plane

If $\vec{F} = \nabla f$ is conservative, we can calculate the integral of $\vec{F}$ over a curve C as



$$\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

It is useful to think of the RHS as an integral over the set $\{A, B\}$, which is the "boundary" of C!

$$f(B) - f(A) = \int_{\{A^-, B^+\}} f$$

where A^-, B^+ indicate "orientations" of the points A and B.

If $\vec{F}$ is conservative, then we saw in the last section that $\oint_L \vec{F}\cdot d\vec{r} = 0$ for any <u>loop</u> L. But often $\vec{F}$ will be non-conservative, and $\oint_L \vec{F}\cdot d\vec{r}$ will represent something interesting, like the amount of fluid flowing into (or out of) the region R enclosed by L.

Viewing L as the <u>boundary</u> of R,

<u>Green's theorem</u> will express

$$\oint_L \vec{F} \cdot d\vec{r} = \iint_R G \, dA$$

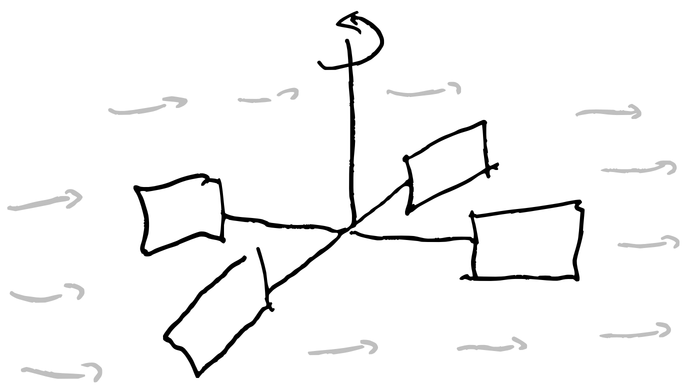
for a <u>function</u> G which is (in an appropriate sense) the "derivative" of $\vec{F}$.

So let $\vec{F} = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$

be a vector field in the plane. For intuition, it is easiest to imagine $\vec{F}$ as the velocity field of some fluid.

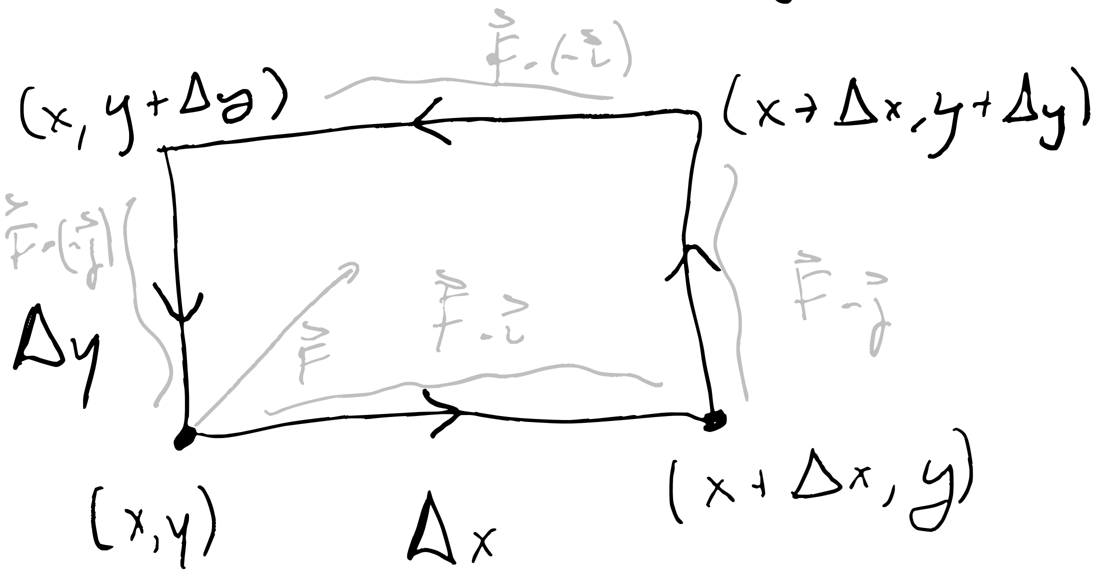
We will define a notion of

circulation density that measures how a "paddle wheel" placed in the fluid (along an axis orthogonal to the plane) will be moved by the fluid.



ex: $\vec{F} = \vec{\imath}$ will cause a paddle-wheel to rotate counterclockwise (viewed from above)

Consider a small rectangle of width Δx and height Δy:



Circulation, or flow, has units $\frac{\text{area}}{\text{time}}$. The flow rates on each segment of the boundary are:

$\downarrow$

Bottom: $\vec{F}(x,y)\cdot\vec{i}\,\Delta x = M(x,y)\,\Delta x$

Top: $\vec{F}(x,y+\Delta y)\cdot(-\vec{i})\,\Delta x = -M(x,y+\Delta y)\,\Delta x$

Left: $\vec{F}(x,y)\cdot(-\vec{j})\,\Delta y = -N(x,y)\,\Delta y$

Right: $\vec{F}(x+\Delta x,y)\cdot\vec{j}\,\Delta y = N(x+\Delta x,y)\,\Delta y$

Adding opposite pairs:

$$\text{Top} + \text{Bottom} = -\left(M(x,y+\Delta y) - M(x,y)\right)\Delta x$$

$$\approx -\frac{\partial M}{\partial y}\,\Delta y\,\Delta x$$

$$\text{Left} + \text{Right} = \left(N(x+\Delta x,y) - N(x,y)\right)\Delta y$$

$$\approx \frac{\partial N}{\partial x}\,\Delta x\,\Delta y$$

The net circulation around the boundary (relative to the CCW orientation) is the sum of these!

$$\text{Circulation rate around rectangle} \approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$

We get the <u>circulation density</u> ("circulation per unit area") by dividing by the area of the rectangle.

$$\text{Circulation density} = \frac{\text{circulation rate}}{\Delta x \Delta y}$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Def The <u>circulation density</u> of a vector field

$$\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$$

at the point (x,y) is the (scalar!) expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

This is also called the <u>$\vec{k}$-component of curl</u>, written $(\nabla \times \vec{F}) \cdot \vec{k}$ or $(\text{curl } \vec{F}) \cdot \vec{k}$, since

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

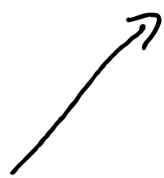
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & 0 \end{vmatrix}$$

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \vec{k}.$$

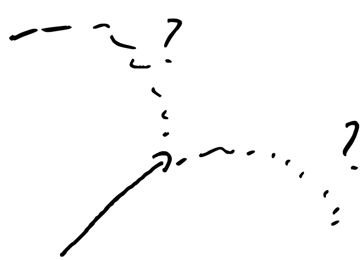
If the circulation density at a point is >0, a paddle-wheel placed there will rotate <u>counter-clockwise</u>. If the circulation density is <0, it will rotate

<u>clockwise</u>. Let's think about why this is:

Suppose $\vec{F}$ is pointing like so:



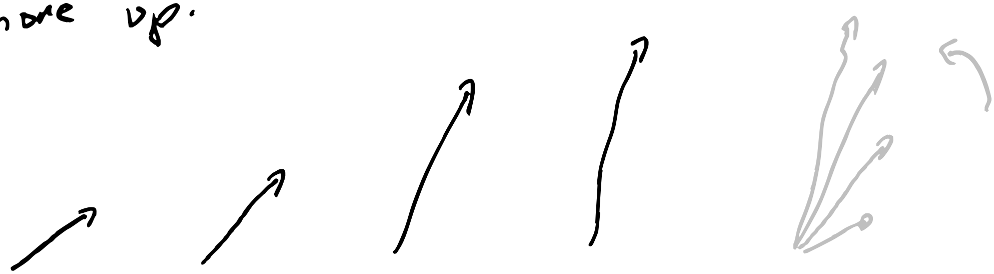
It could be moving CCW or CW:



Circ density $= \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$, so let's think about what each of

What do $\frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ mean.

If $\frac{\partial N}{\partial x} > 0$, then as we move right (increasing x), $\vec{F}$ will point more up!



If $\frac{\partial M}{\partial y} > 0$, then as we move up (increasing y), $\vec{F}$ will point more right!

So $\frac{\partial N}{\partial x}$ contributes to CCW motion, $\frac{\partial M}{\partial y}$ contributes to CW motion, and

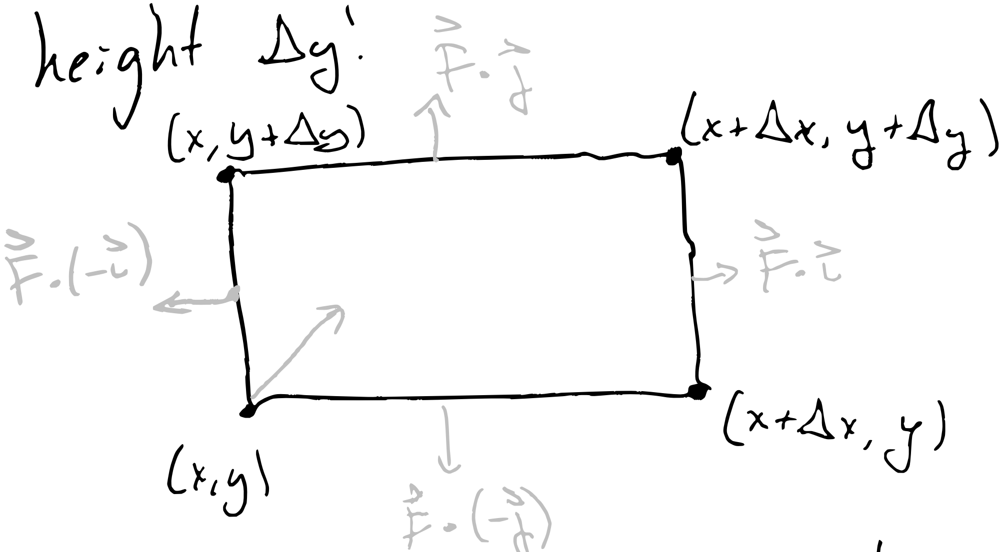
$$\text{circ density} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

tells us which "wins out".

If we use normal vectors instead of tangent vectors in the definition of circulation density, we get the concept of <u>flux density</u> or <u>divergence</u>.

This will describe how a fluid is flowing out of (as opposed to circulating around) a rectangle, and measures the extent to which a fluid is expanding or compressing at a point. For example, the divergence of a liquid's velocity field is always 0, since liquids are incompressible; but divergence of a gas's velocity field can be very interesting.

We'll repeat our analysis of a small rectangle of width Δx and height Δy:



The approximate outward flow along each edge of the boundary is

Bottom: $\vec{F}(x,y)\cdot(-\vec{j})\,\Delta x = -N(x,y)\,\Delta x$

Top: $\vec{F}(x,y+\Delta y)\cdot\vec{j}\,\Delta x = N(x,y+\Delta y)\,\Delta x$

Left: $\vec{F}(x,y)\cdot(-\vec{i})\,\Delta y = -M(x,y)\,\Delta y$

Right: $\vec{F}(x+\Delta x,y)\cdot\vec{i}\,\Delta y = M(x+\Delta x,y)\,\Delta y$

Summing these gives

$$\text{Top} + \text{Bottom} = \big(N(x, y+\Delta y) - N(x,y)\big)\Delta x$$

$$\approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

$$\text{Left} + \text{Right} = \big(M(x+\Delta x, y) - M(x,y)\big)\Delta y$$

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$

So the <u>total flux</u> along the boundary is approximately

$$\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\,\Delta y,$$

and the <u>flux density</u> (flux per unit area) is $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$.

Def'n The <u>divergence</u> (flux density) of a vector field $\vec{F} = M(x,y)\vec{i} + N(x,y)\vec{j}$ at the point (x,y) is the expression

$$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

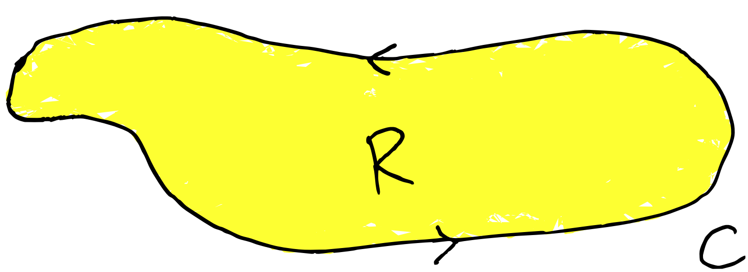
The divergence is also denoted $\nabla \cdot \vec{F}$:

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}\right) \cdot \left(M\vec{i} + N\vec{j}\right)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

Green's theorem

Now we can state Green's theorem. First, recall that a curve C is <u>simple</u> if it never crosses itself. A simple closed curve C encloses a region R.



Such a curve can be traversed <u>counterclockwise</u> (R is always to your left) or clockwise (R is always to your right).

If we change the orientation (i.e. the direction we traverse) C, the line integral $\oint_C \vec{F} \cdot d\vec{r}$ will change sign. We will always take the counterclockwise orientation of C unless explicitly indicated otherwise.

Now we state Green's theorem, in two forms. Let's get our hypotheses out of the way first:

C is a piecewise smooth simple closed curve, enclosing a region R. $\vec{F} = M\vec{i} + N\vec{j}$ is a vector field defined on an open region containing R, such that M and N have continuous first ∂'s.

<u>Green's thm (Circulation - Curl form)</u>

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_R (\text{curl}\, \vec{F} \cdot \vec{k})\, dA$$

<u>Green's thm (Flux - Divergence form)</u>

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_R \text{div}\, \vec{F}\, dA$$

More explicitly,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

CCW circulation curl integral

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

outward flux divergence integral

We might also call the first form the <u>tangential</u> form and the second the <u>normal</u> form.

We'll omit the proof of Green's theorem (see the textbook for a proof in some special cases).

Instead, let's do some

Examples

ex) Find the counterclockwise circulation and outward flux for the field $\vec{F} = 2xy\,\hat{\imath} - y\,\hat{\jmath}$ over the unit square $0 \le x, y \le 1$.

$\int$

circulation $= \oint 2xy\,dx - y\,dy$

$$= \int_0^1\int_0^1 -2y\,dx\,dy$$

$$= -1$$

flux $= \oint 2xy\,dy + y\,dx$

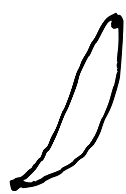
$$= \int_0^1\int_0^1 (2y-1)\,dx\,dy$$

$$= \int_0^1 (2y-1)\,dy = -\frac{1}{2}$$

ex] Find the counterclockwise circulation and outward flux of the field

$$\vec{F} = (x-y)\vec{i} + (2x+y)\vec{j}$$

over the region enclosed by $y = x^2$, $0 \le x \le 1$ and $y = \sin\left(\frac{\pi}{2}x\right)$, $0 \le x \le 1$.



$$\text{circulation} = \oint_C (x-y)\,dx + (2x+y)\,dy$$

$$= \int_0^1 \int_{x^2}^{\sin(\pi x/2)} (2+1)\,dy\,dx$$

$$= 3\int_0^1 \left(\sin\left(\frac{\pi x}{2}\right) - x^2\right) dx$$

$$= 3\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}x^3\right]_0^1$$

$$= \frac{6}{\pi} - 1$$

$$\text{flux} = \oint_C (x-y)\,dy - (2x+y)\,dx$$

$$= \int_0^1 \int_{x^2}^{\sin(\pi x/2)} (1+1)\,dy\,dx$$

$$= \frac{4}{\pi} - \frac{2}{3}$$