§16.5 Surfaces and Area

Parametric surfaces

If S is a region in space, there are two ways we can describe S: via an implicit description or via a parametric description.

An implicit description of S, is a set of equations which are satisfied by all the points in S, and only the points in S. For example, an implicit description of a circle of radius a centered at the origin (in $\mathbb{R}^2$) is

$$x^2 + y^2 = a^2.$$

A <u>parametric</u> description of S is a function whose range lies in S. Ideally the range will be all of S, but we can also work with several different parametrizations such that each point of S is in the range of at least one parametrization. For example, a parametric description of the circle of radius a is

$$\vec{r}(\theta) = (a\cos\theta,\ a\sin\theta) \qquad 0 \le \theta \le 2\pi.$$

Both types of description are valuable: an implicit description tells us how to check if a given point (x_0, y_0, z_0) is in S, while a parametric description tells us how to produce examples of points in S. If we can isolate one of the variables, then we get the best of both worlds and call it an explicit form for S; this is only possible if S is the graph of a function. Both producing points on S starting from an implicit description, or checking if $(x_0, y_0, z_0) \in S$ starting from a parametric description requires us to solve systems of equations, which may be very difficult.

"Solving equations" can be interpreted, in a general sense, as finding a parametric description of a region given an implicit one.

	Implicit	Parametric	Explicit
Form	$F(x,y)=0$	$x=f(t)$ $y=g(t)$	$y=f(x)$ or $x=g(y)$
Easy to check if $P\in S$	✓	X	✓
Easy to produce $P\in S$	X	✓	✓

Note that an explicit equation $y=f(x)$ can be written as an implicit equation

$$y-f(x)=0$$

or as a parametric equation

$$x=t,\quad y=f(t).$$

Roughly speaking, k equations in n variables will determine an $(n-k)$-dimensional shape (i.e. "each constraint removes one degree of freedom"), while a function with k inputs and n outputs, $k \leq n$, will have a k-dimensional range.

	Implicit	Parametric
curve in 2d	$F(x,y)=0$	$\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}$
curve in 3d	$F_1(x,y,z)=0$ $F_2(x,y,z)=0$	$\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}$ $+z(t)\vec{k}$
surface in 3d	$F(x,y,z)=0$	$\vec{r}(u,v)=x(u,v)\vec{i}$ $+y(u,v)\vec{j}$ $+z(u,v)\vec{k}$

(This is not quite true – the functions need to be "non-redundant", in the sense that the columns of the Jacobian matrix need to be "linearly independent". This is true "almost all of the time", so we won't dwell on it.

We are interested in the specific case of surfaces in 3d space.

We consider functions

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

for (u,v) contained in a subspace

R (called the <u>parameter domain</u>) of the plane. We will assume that $\vec{r}$ is one-to-one on the interior of R. Usually we will take R to be a rectangle

$a \le u \le b, \quad c \le v \le d.$

<u>Examples</u>

① Find a parametrization of the cone $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$.

Without the restriction on z,

this would already be giving us an explicit description of the cone. But the parameter domain would then be a disk, whereas we'd like it to be a rectangle. So we use polar coordinates for the disk to get

$$x = u\cos(v) \qquad 0 \le u \le 1$$

$$y = u\sin(v) \qquad 0 \le v \le 2\pi$$

$$z = u$$

(2) Here are three different parametrizations of the unit

sphere in 3d space,

$$x^2 + y^2 + z^2 = 1.$$

a) "Cartesian": writing

$$z = \sqrt{1 - x^2 - y^2},$$

we can take

$$x = u\cos(v) \qquad 0 \le u \le 1$$

$$y = u\sin(v) \qquad 0 \le v \le 2\pi$$

$$z = \sqrt{1 - u^2}$$

Another option is

$x = u$ $\qquad -1 \le u \le 1$

$y = v\sqrt{1-u^2}$ $\qquad -1 \le v \le 1$

$z = \sqrt{(1-u^2)(1-v^2)}$

b) "Spherical": as we've already seen,

$x = \cos(u)\sin(v)$ $\qquad 0 \le u \le 2\pi$

$y = \sin(u)\sin(v)$ $\qquad 0 \le v \le \pi$

$z = \cos(v)$

c) "Stereographic": let's start with a 2d version.

Suppose we wanted to parametrize the unit <u>circle</u> by the <u>slope</u> of (the line segment from the origin to) a point $(\cos\theta, \sin\theta)$; then we would take $t = \tan\theta$. However, we already know this won't work, as it can't distinguish θ from $\theta + \pi$. We can fix this by instead taking $\boxed{t = \tan\frac{\theta}{2}}$

If we do this, then the half-angle identities give

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$

$$\Rightarrow \quad t = \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

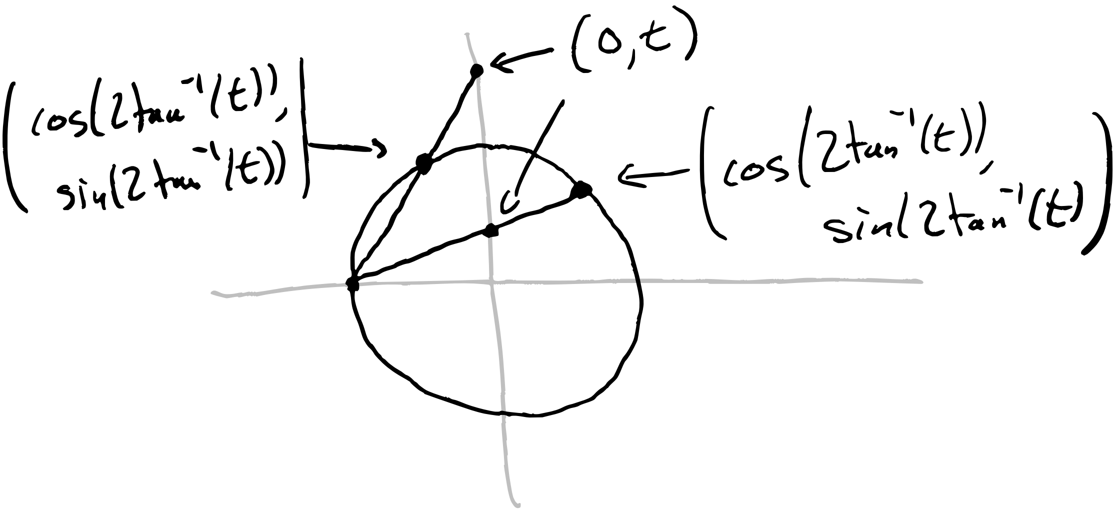
$$= \sqrt{\frac{1-\cos^2\theta}{|1+\cos\theta|^2}}$$

$$= \frac{\sin\theta}{1+\cos\theta}$$

or similarly

$$t = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta}$$

The formula $t = \frac{\sin\theta}{1+\cos\theta}$ can be interpreted as saying that t is the slope of the line connecting $\underline{(-1,0)}$ (instead of $(0,0)$) to $(\cos\theta, \sin\theta)$; this line passes through $(0,t)$.



We need to find $(\cos\theta, \sin\theta)$ in terms of t:

$$t = \frac{1-\cos\theta}{\sin\theta}$$

$$\Rightarrow 1+t^2 = \frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{\sin^2\theta}$$

$$= \frac{2}{\sin\theta}\,\frac{1-\cos\theta}{\sin\theta}$$

$$\Rightarrow \sin\theta = \frac{2t}{1+t^2}$$

Similarly,

?

$$t = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow 1+t^2 = \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{(1+\cos\theta)^2}$$

$$= \frac{2}{1+\cos\theta}$$

$$\Rightarrow \cos\theta = \frac{2}{1+t^2} - 1$$

$$= \frac{1-t^2}{1+t^2}$$

So our parametrization is

$$(\cos\theta, \sin\theta) = \frac{1}{1+t^2}(1-t^2, 2t)$$

$$\boxed{-\infty < t < \infty}$$

This doesn't allow us to represent the point $(-1,0)$. But if we rewrite

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

$$= \left(\frac{1/t^2 - 1}{1/t^2+1}, \frac{2/t}{1/t^2+1}\right) \quad \text{for } t \neq 0$$

$$\to \left(\frac{-1}{1}, \frac{0}{1}\right) = (-1,0) \quad \text{as } t \to \pm\infty$$

Then we can represent $(-1,0)$ by $t = \infty$ (this is an "unsigned" infinity).

Rmk The substitution

$$t = \tan\frac{\theta}{2}, \quad \cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}$$

is sometimes called the "universal trig substitution" or the "Weierstrass substitution". It is very handy for evaluating trigonometric integrals.

Now we want a 3d version of this. Given any point $P(x, y, z)$ on the unit sphere other than the "north pole" $N(0,0,1)$, we consider the line

$$L: \quad N + t(P - N)$$

$$= (tx, ty, 1 + tz - t)$$

passing through N and P. This intersects the xy-plane at some point $(u, v, 0)$, and we call (u, v) the **stereographic coordinates** of P (relative to N — we could just as well projected from the south pole, or any point on the sphere).

To find the conversion formulas, we need to solve for when the z-component of L is 0!

$$1 + tz - t = 0$$

$$\Rightarrow t = \frac{1}{1-z}$$

note this isn't defined at N!

So $\boxed{u = \frac{x}{1-z}, \quad v = \frac{y}{1-z}}$

To go the other way, we consider the line L passing through $N(0,0,1)$ and $Q(u,v,0)$, then solve for when this is on the sphere!

$$L: N + t(Q - N)$$

$$= (tu, tv, 1-t)$$

$$(tu)^2 + (tv)^2 + (1-t)^2 = 1$$

$$\Rightarrow t^2(u^2+v^2) + (1-2t+t^2) = 1$$

$$\Rightarrow t = \frac{2}{1+u^2+v^2}$$

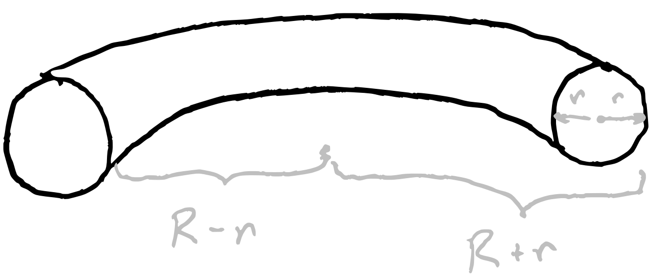
So

$$\boxed{(x,y,z) = \frac{1}{1+u^2+v^2}(2u, 2v, u^2+v^2-1)}$$

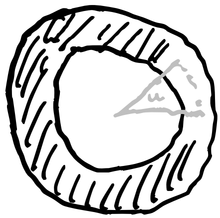
Stereographic coordinates have the interesting property that any <u>line</u>

in the uv-plane gets transformed into a circle on the surface of the sphere.

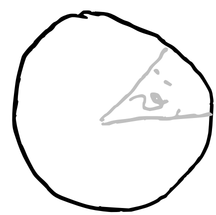
(3) Find a parametrization of the torus with major radius R and minor radius r.



We will a parameter u for the angle on the "wide circle" and v for the angle on the "small circle".



view from above



"slice" view

We should start with $(R\cos(u), R\sin(u), 0)$.

We can also see that the z component will be $r\sin(v)$.

We will also move $r\cos(v)$ along the line from the origin to $(R\cos(u), R\sin(v), 0)$. Our parametrization is thus

$$x = (R + r\cos(v))\cos(u)$$

$$y = (R + r\cos(v))\sin(u)$$

$$z = r\sin(v)$$

$$0 \le u \le 2\pi$$
$$0 \le v \le 2\pi$$

Surface area

Consider a surface S parametrized by

$$\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$$

$$a \le u \le b$$
$$c \le v \le d$$

At any point of S, the partial derivatives

$$\vec{r}_u = \frac{\partial f}{\partial u}\vec{i} + \frac{\partial g}{\partial u}\vec{j} + \frac{\partial h}{\partial u}\vec{k}$$

$$\vec{r}_v = \frac{\partial f}{\partial v}\vec{i} + \frac{\partial g}{\partial v}\vec{j} + \frac{\partial h}{\partial v}\vec{k}$$

are <u>tangent</u> to the surface S.
Provided they are not parallel, they give a <u>parametric</u> description of the tangent plane to S:

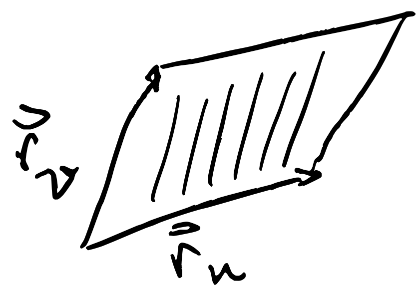
Prop Let $\vec{r}(u,v)$ be a parametrization of a surface S. If $\vec{r}_u(u_0, v_0)$ is not parallel to $\vec{r}_v(u_0, v_0)$, then the tangent plane to S at $P = \vec{r}(u_0, v_0)$ is given by

$$\boxed{P + s\,\vec{r}_u(u_0,v_0) + t\,\vec{r}_v(u_0,v_0) \qquad s,t \in \mathbb{R}}$$

Recall that two vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if their cross product $\vec{a} \times \vec{b}$ is $\vec{0}$.

Def A surface S parametrized by $\vec{r}(u,v)$ is smooth if $\vec{r}_u$ and $\vec{r}_v$ is continuous, and $\vec{r}_u \times \vec{r}_v \neq 0$ on the interior of the parameter domain.

In this case, the vector $\vec{r}_u \times \vec{r}_v$ will point in the normal direction of the surface S. The magnitude $|\vec{r}_u \times \vec{r}_v|$ is the <u>area</u> of the parallelogram spanned by the two vectors!



$$A = |\vec{r}_u \times \vec{r}_v|$$

We can thus use this to compute the <u>surface area</u> of S!

$\int$

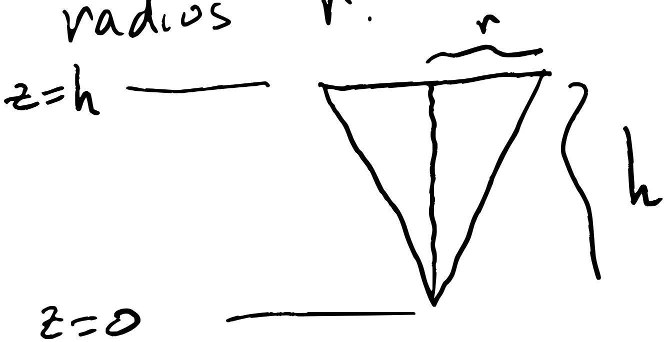
Def Let S be a smooth surface parametrized by $\vec{r}(u,v)$, $a \leq u \leq b$, $c \leq v \leq d$. The **surface area** of S is

$$A = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

We sometimes write $d\sigma = |\vec{r}_u \times \vec{r}_v| \, du \, dv$ for the "surface area differential".

<u>ex</u> Find the surface area of a cone of height h and base radius r.



(flipping the cone makes the formulas a little easier)

We use the parametrization

$$x = \frac{r}{h} z \cos(\theta)$$

$$y = \frac{r}{h} z \sin(\theta) \qquad 0 \le \theta \le 2\pi$$

$$z = z \qquad 0 \le z \le h$$

Let's write $\vec{c}(\theta, z)$ instead of $\vec{r}(\theta, z)$ to avoid confusion with r.

Then $\vec{c}_\theta \times \vec{c}_z$ is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{r}{h} z \sin\theta & \frac{r}{h} z \cos\theta & 0 \\ \frac{r}{h}\cos\theta & \frac{r}{h}\sin\theta & 1 \end{vmatrix}$$

$$= \frac{rz}{h}\left(\cos\theta \, \vec{i} + \sin\theta \, \vec{j} - \frac{r}{h} \vec{k} \right)$$

so $$\left| \vec{c}_\theta \times \vec{c}_z \right| = \frac{rz}{h^2} \sqrt{h^2 + r^2}$$

The surface area is thus

$$A = \iint_R \left| \vec{c}_\theta \times \vec{c}_z \right| \, d\theta \, dz$$

$$= \frac{r\sqrt{h^2+r^2}}{h^2} \int_0^h \int_0^{2\pi} z \, d\theta \, dz$$

$$= \pi r \sqrt{h^2+r^2}$$

$\boxed{!}$ This is only calculating the surface area of the <u>sides</u> of the cone. If you also want to count the surface area of the base of the cone, you should add πr^2.

ex Find the surface area of a sphere of radius r.

We'll use spherical coordinates:

$$x = r\cos\theta \sin\phi \qquad 0 \leq 2\pi \leq \theta$$

$$y = r\sin\theta \sin\phi \qquad 0 \leq \phi \leq \pi$$

$$z = r\cos\phi$$

Again let's write $\vec{s}(\theta, \phi)$

instead of $\vec{r}(\theta, \phi)$. Then

$\vec{s}_\theta \times \vec{s}_\phi =$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta\sin\phi & r\cos\theta\sin\phi & 0 \\ r\cos\theta\cos\phi & r\sin\theta\cos\phi & -r\sin\phi \end{vmatrix}$$

$$= r^2 \sin\phi \left(-\cos\theta \sin\phi\, \vec{i} - \sin\theta\sin\phi\, \vec{j} - \cos\phi\, \vec{k} \right)$$

So $\left| \vec{s}_\theta \times \vec{s}_\phi \right| = r^2 \sin\phi$

The surface area is thus

$$A = \iint |\vec{s}_\theta \times \vec{s}_\phi| \, d\theta \, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} r^2 \sin\phi \, d\theta \, d\phi$$

$$= 2\pi r^2 \int_0^{\pi} \sin\phi \, d\phi$$

$$= 4\pi r^2$$

ex Find the surface area of a torus of major radius R and minor radius r.

Recall we found the parametrization

$$\left.\begin{aligned} x &= (R + r\cos\phi)\cos\theta \\ y &= (R + r\cos\phi)\sin\theta \\ z &= r\sin\phi \end{aligned}\right\} \begin{aligned} 0 \le \theta \le 2\pi \\ 0 \le \phi \le 2\pi \end{aligned}$$

Call this $\vec{t}(\theta, \phi)$. Then

$$|\vec{t}_\theta \times \vec{t}_\phi| =$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(R+r\cos\phi)\sin\theta & (R+r\cos\phi)\cos\theta & 0 \\ -r\sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi \end{vmatrix}$$

$$= (R+r\cos\phi)r\left(\cos\theta\cos\phi\,\vec{i} + \sin\theta\cos\phi\,\vec{j} + \sin\phi\,\vec{k}\right)$$

so

$$|\vec{t}_\theta \times \vec{t}_\phi| = (R+r\cos\phi)r.$$

The surface area is thus

$$A = \iint |\vec{t}_\theta \times \vec{t}_\phi| \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_0^{2\pi} (R + r\cos\phi)\, r \, d\theta \, d\phi$$

$$= 4\pi^2 R r$$

$\int$

Surface area of implicit surfaces

Of course, it is not always easy to find a parametrization of a surface. How can we find the surface area of a surface S given to us *implicitly*, as a level set

$$F(x,y,z) = c \text{ ?}$$

The basic idea is that we will compute the area of S from the area of its *projection* onto a plane P. So let $\vec{p}$ be a unit vector normal to a plane

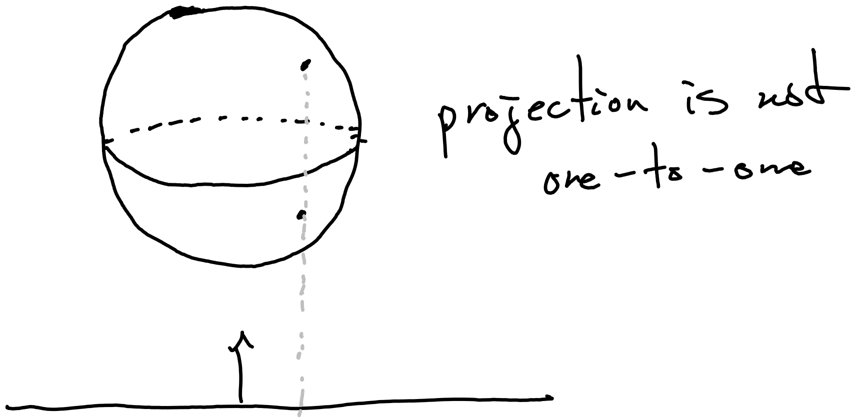
P; we will usually take $\vec{p} = \vec{k}$, so that P is the xy-plane, but sometimes it is useful to take $\vec{p} = \vec{i}$ or $\vec{p} = \vec{j}$ or something else.

We will assume that

$$\nabla F(x,y,z) \cdot \vec{p} \neq 0$$

for all (x,y,z) with $F(x,y,z) = c$, which guarantees that the projection from S to P is one-to-one.

(For example, if S were the unit sphere $x^2+y^2+z^2=1$, this is not possible for <u>any</u> choice of $\vec{p}$.



To calculate the surface area of a sphere using the method we're about to present, you'd need to calculate the area of a <u>hemisphere</u> and then double that).

So let $\vec{p} = \vec{k}$ (to simplify exposition), and suppose that

$\nabla F \cdot \vec{p} = \nabla F \cdot \vec{k} = F_z \neq 0$. The Implicit Function Theorem then tells us that $S = \{(x,y,z) : F(x,y,z) = c\}$ is the graph

$$z = h(x,y)$$

of some (unknown) function $h(x,y)$.

This means that

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + h(x,y)\vec{k}$$

is a <u>parametric</u> description of S, and we have

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial h}{\partial x} \\ 0 & 1 & \frac{\partial h}{\partial y} \end{vmatrix}$$

$$= -\frac{\partial h}{\partial x}\vec{i} - \frac{\partial h}{\partial y}\vec{j} + \vec{k}$$

This is not so useful as we don't have access to $h(x,y)$. However, we can find $\partial h/\partial x$ and $\partial h/\partial y$ by implicit differentiation!

!

$$F(\vec{r}(x,y)) = 0$$

$$\Rightarrow \nabla F \cdot \vec{r}_x = 0 \quad \text{and} \quad \nabla F \cdot \vec{r}_y = 0$$

$$\Rightarrow F_x + F_z \frac{\partial h}{\partial x} = 0 \Rightarrow \frac{\partial h}{\partial x} = -\frac{F_x}{F_z}$$

$$F_y + F_z \frac{\partial h}{\partial y} = 0 \Rightarrow \frac{\partial h}{\partial y} = -\frac{F_y}{F_z}$$

So

$$\vec{r}_x \times \vec{r}_y = \frac{1}{F_z}\left(F_x \vec{i} + F_y \vec{j} + F_z \vec{k}\right)$$

$$= \frac{\nabla F}{\nabla F \cdot \vec{k}}$$

This gives

<u>Thm</u> (Surface area of an implicit surface)

The area of a surface $F(x,y,z)=c$ over a plane region R is

$$A = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}\, dA$$

where $\nabla F \cdot \vec{p} \neq 0$.

<u>ex</u> Find the surface area of the band cut from the paraboloid

$$z = x^2 + y^2$$

by the planes $z=1$ and $z=4$.

The region of integration R is

$$1 \le x^2 + y^2 \le 4,$$

i.e. an annulus of inner radius 1 and outer radius 2. We can parametrize this by

$$x = r\cos\theta \qquad 1 \le r \le 2$$

$$y = r\sin\theta \qquad 0 \le \theta \le 2\pi$$

Let $F(x,y,z) = x^2 + y^2 - z$, so

$$\nabla F = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$= 2r\cos\theta\,\vec{i} + 2r\sin\theta\,\vec{j} - \vec{k}$$

and

$$\frac{|\nabla F|}{|\nabla F \cdot \vec{k}|} = \sqrt{1+4r^2}$$

Thus

$$A = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \vec{k}|}\,dx\,dy$$

$$= \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\,(r\,dr\,d\theta)$$

$$= \frac{1}{8}\int_0^{2\pi}\int_5^{17} \sqrt{u}\, du\, d\theta$$

$u = 1+4r^2$

$du = 8r\,dr$

$$= \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_5^{17} = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

<u>ex</u> Find a formula for the surface area of a graph

$$z = f(x,y)$$

over a region R.

We parametrize the graph by

$[$

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$$

Then

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

$$= -f_x\vec{i} - f_y\vec{j} + \vec{k}$$

So

$$A = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

We could also have approached this

by considering the function

$$F(x,y,z) = z - f(x,y)$$

so that

$$\nabla F = -f_x \vec{i} - f_y \vec{j} + \vec{k}$$

and thus

$$\frac{|\nabla F|}{|\nabla F \cdot \vec{k}|} = \sqrt{1 + f_x^2 + f_y^2}$$

But this is a little silly, since we derived the formula

$$|\vec{r}_x \times \vec{r}_y| = \frac{|\nabla F|}{|\nabla F \cdot \vec{k}|}$$ by doing the above calculation