§16.6 Surface integrals

Just as we did for curves, we can integrate scalar functions $G(x,y,z)$ over a surface S,

$$\iint_S G(x,y,z)\, d\sigma$$

When $G(x,y,z)=1$, we recover the

surface area of S. If $G(x,y,z)$ is a density function (mass per unit area), then the integral represents the <u>mass</u> of (the physical object with shape) S.

<u>Def</u> Let $G(x,y,z)$ be a continuous function defined on a (piecewise) smooth surface S.

a) If S is given <u>parametrically</u> by $\vec{r}(u,v)$, $(u,v) \in R$, then

$$\iint_S G\,d\sigma = \iint_R G(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

b) If S is given <u>implicitly</u> as

$S = \{(x,y,z) : F(x,y,z) = c\}$, then

$$\iint_S G(x,y,z)\, d\sigma = \iint_R G(x,y,z) \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}\, dA$$

where $\vec{p}$ is a unit vector normal to the plane region R, and $\nabla F \cdot \vec{p} \neq 0$ on S.

c) If S is given <u>explicitly</u> as $z = f(x,y)$, then

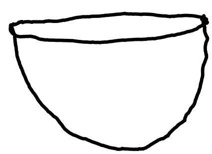
$$\iint_S G\, d\sigma = \iint_R G(x,y,f(x,y)) \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

ex| Find the mass of a bowl in the shape of a hemisphere

$$x^2 + y^2 + z^2 = a^2, \quad z \le 0$$

if the density is

$$\delta(x,y,z) = 1 - z$$



We parametrize the bowl using spherical coordinates:

$$\vec{r}(\theta,\phi) \begin{cases} x = a\cos\theta\sin\phi \\ y = a\sin\theta\sin\phi \\ z = a\cos\phi \end{cases} \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ \frac{\pi}{2} \le \phi \le \pi \end{array}$$

Recall from previous calculations that $|\vec{r}_\theta \times \vec{r}_\phi| = a^2 \sin\phi$. Thus

$$\text{Mass} = \iint_S \delta \, d\sigma$$

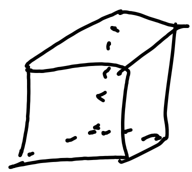
$$= \int_0^{2\pi}\int_{\pi/2}^{\pi} (1 - a\cos\phi)(a^2 \sin\phi) \, d\phi \, d\theta$$

$$= 2\pi a^2 \int_{\pi/2}^{\pi} \sin\phi \, d\phi - 2\pi a^3 \int_{\pi/2}^{\pi} \cos\phi \sin\phi \, d\phi$$

$$= 2\pi a^2 \Big[-\cos\phi\Big]_{\pi/2}^{\pi} + \pi a^3 \left[\frac{\cos 2\phi}{2}\right]_{\pi/2}^{\pi}$$

$$= 2\pi a^2 + \pi a^3 = \pi a^2 (2 + a).$$

ex) Integrate $G(x,y,z) = xyz$ over the unit cube C



C is not smooth, but it is piecewise smooth, since each of the six faces is smooth. So

$$\iint_C = \iint_{\text{Front}} + \iint_{\text{Back}} + \iint_{\text{Left}} + \iint_{\text{Right}} + \iint_{\text{Top}} + \iint_{\text{Bottom}}$$

In the case at hand, note that

$$\iint_{\text{Left}} xyz\,d\sigma = \iint_{\text{Bottom}} xyz\,d\sigma = \iint_{\text{Front}} xyz\,d\sigma$$

$= 0$ since $x = 0$ on Left,
$z = 0$ on Bottom,
$y = 0$ on Front.

So

$$\iint_{C} xyz\,d\sigma = \iint_{\text{Back}} xyz\,d\sigma + \iint_{\text{Top}} xyz\,d\sigma + \iint_{\text{Right}} xyz\,d\sigma$$

$$= 3\iint_{\text{Top}} xyz\,d\sigma \text{ by symmetry.}$$

The top face is the graph of $z=1$ over $0 \le x, y \le 1$, so we get

$$\iint_C xyz \, d\sigma = 3 \iint_{Top} xyz \, d\sigma$$

$$= 3 \int_0^1 \int_0^1 xy \, dx \, dy$$

$$= \frac{3}{2} \int_0^1 y \, dy = \frac{3}{4}$$

Orientations

Recall that a simple closed curve admits two orientations:
counterclockwise or clockwise,

and that the line integral $\oint_C \vec{F} \cdot d\vec{r}$
changes sign (but has the same absolute value) if we reverse the orientation.

There is also a notion of orientation for surfaces, which distinguishes "inside/outside" rather than "clockwise/counterclockwise"

For instance, if you are holding a spherical object, you're more interested in the outside, but if you're standing (floating?) in a spherical chamber, the "inside" is more relevant.

Formally, an <u>orientation</u> of a surface S is a continuous choice of unit normal vector,

$\vec{n}(x,y,z)$ (defined for $(x,y,z) \in S$)

It is equivalent to ask for a non-vanishing normal vector $\vec{N}(x,y,z)$, since we can then get a unit normal vector by dividing by the length:

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|}.$$

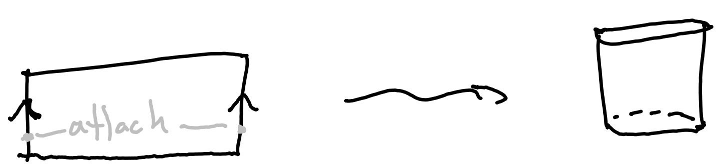
A surface is called orientable if it admits an orientation; an oriented surface consists of a surface together with a choice of orientation.

(Note the difference between orientable and oriented.) A connected orientable surface admits exactly two orientations: if $\vec{n}$ is one, the other is $-\vec{n}$.

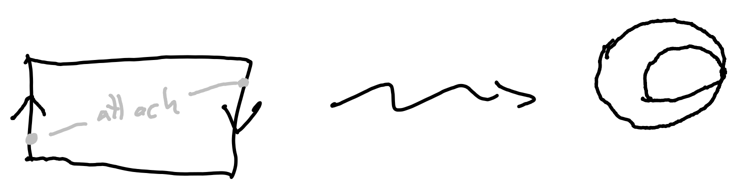
(In general, a surface with k connected components admits 2^k orientations, since we have two on each component.)

Any surface which is the boundary of a region in space is orientable, since we can then take $\vec{n}$ to point "away" from the region.

However, not every surface is orientable. The standard example is a Möbius strip. You can make a Möbius strip by taking a small strip of paper and attaching the ends together after a twist.



cylinder

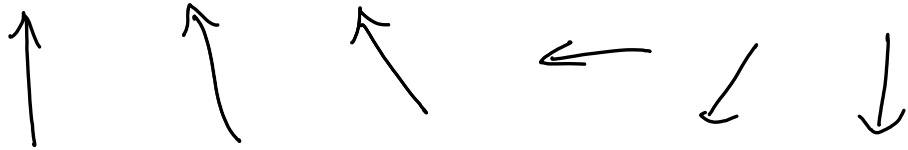


Möbius strip

You will see that it is "one-sided". Let's approach this mathematically: we'll start with a parametrization of a cylinder,

$$x = \cos\theta \qquad 0 \le \theta \le 2\pi$$
$$y = \sin\theta \qquad -1 \le t \le 1$$
$$z = t$$

We think of this as: draw a circle in (x,y), and at each point attach a straight vertical line. To get a Möbius strip, we want the line ~~to~~ instead twist as we go around the circle.



so instead of

$$\vec{r}(\theta,t) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + t\,\vec{k}$$

we take

$$\vec{r}(\theta,t) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$
$$+ t\left(\sin\frac{\theta}{2}\left(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}\right) + \cos\frac{\theta}{2}\,\vec{k}\right)$$

$$= \cos\theta\left(1 + t\sin\frac{\theta}{2}\right)\vec{i}$$
$$+ \sin\theta\left(1 + t\sin\frac{\theta}{2}\right)\vec{j}$$
$$+ t\cos\frac{\theta}{2}\,\vec{k}$$

If you are brave/masochistic, you could compute $\vec{r}_\theta \times \vec{r}_t$ and see that it vanishes on the interior of the parameter domain.

<u>Warning</u> In <u>Avengers: Endgame</u>, Tony Stark uses a Möbius strip to invent time travel. This will <u>not</u> work in reality.

Surface integrals of vector fields

Let $\vec{F}(x,y,z)$ be a continuous vector field, and let S be a smooth surface with orientation $\vec{n}$. We define the integral of $\vec{F}$ over S to be

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma.$$

This is also called the flux of $\vec{F}$ over S. If $\vec{F}$ is the velocity field of a fluid in 3d space, then $\iint_S \vec{F} \cdot \hat{n}\, d\sigma$ is the net rate at which the fluid is crossing S per unit time, in the direction specified by $\hat{n}$.

The above formula can be simplified slightly. If S is given parametrically by $\vec{r}(u,v)$, then we can take the orientation to be

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

which gives

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma$$

$$= \iint_R \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

$$= \iint_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv$$

so we don't need to compute $|\vec{r}_u \times \vec{r}_v|$. Recall that $\vec{F} \cdot (\vec{r}_u \times \vec{r}_v)$

is called the <u>box product</u> of $\vec{F}$, $\vec{r}_u$, and $\vec{r}_v$, and represents the (signed) volume of the box/parallelepiped spanned by $\vec{F}$, $\vec{r}_u$, and $\vec{r}_v$.

<u>ex</u>] Find the flux of

$$\vec{F} = yz\vec{i} - xz\vec{j} - xy\vec{k}$$

through the parabolic cylinder

$$y = x^2, \quad 0 \leq x \leq 1, \quad 0 \leq z \leq 3$$

oriented so that the normal vector points in the negative y direction.

Let

$$\vec{r}(x,z) = x\vec{i} + x^2\vec{j} + z\vec{k},$$

Then

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2x\vec{i} - \vec{j}$$

which gives the desired orientation.

Then

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma$$

$$= \iint_R \vec{F} \cdot (\vec{r}_x \times \vec{r}_z)\, dx\, dz$$

$$= \int_0^3 \int_0^1 \left(x^2 z\,\vec{i} - xz\,\vec{j} - x^3\,\vec{k}\right) \cdot \left(2x\,\vec{i} - \vec{j}\right) dx\,dz$$

$$= \int_0^3 \int_0^1 \left(2x^3 z + xz\right) dx\,dz$$

$$= \int_0^3 z\,dz = \frac{9}{2}$$

Alternatively, suppose that S is given as the level surface $g(x,y,z) = c$ of some function g. Then

We can take our orientation to be $\pm \dfrac{\nabla g}{|\nabla g|}$, so that our formula becomes

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \pm \iint_R \vec{F} \cdot \frac{\nabla g}{|\nabla g|} \frac{|\nabla g|}{|\nabla g \cdot \vec{p}|} \, dA$$

$$= \pm \iint_R \vec{F} \cdot \frac{\nabla g}{|\nabla g \cdot \vec{p}|} \, dA$$

ex| Find the flux of

$$\vec{F} = x^2\vec{i} - yz\vec{k}$$

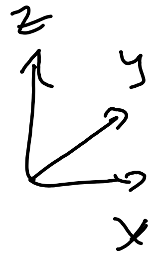
outward through the surface cut from the cylinder

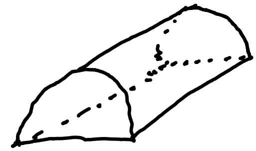
$$x^2 + z^2 = 4, \quad z \geq 0$$

by the planes $y=0$ and $y=3$

Let $g(x,y,z) = x^2+z^2-4$, so

$\nabla g = 2x\vec{i} + 2z\vec{k}$, which indeed points "outward" from the cylinder.





We take our region of integration to be $-2 \le x \le 2,\ 0 \le y \le 3.$

Then

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_R \vec{F} \cdot \frac{\nabla g}{|\nabla g \cdot \vec{k}|}\, dA$$

$$= \int_0^3 \int_{-2}^{2} \left(x^2 \vec{i} - y\sqrt{4-x^2}\,\vec{k}\right) \cdot \frac{\left(2x\vec{i} + 2\sqrt{4-x^2}\,\vec{k}\right)}{2\sqrt{4-x^2}}\, dx\, dy$$

$$= \int_0^3 \int_{-2}^2 \frac{x^3}{\sqrt{4-x^2}} - y\sqrt{4-x^2}\,dx\,dy$$

odd function so $\int_{-2}^{2} = 0$

$$= \underbrace{\left(-\int_0^3 y\,dy\right)}_{=-9/2} \underbrace{\left(\int_{-2}^2 \sqrt{4-x^2}\,dx\right)}_{\text{area of semicircle of radius 2, so } \frac{\pi \cdot 2^2}{2} = 2\pi}$$

$$= -9\pi.$$

Masses and moments

We can model thin sheets/shells of material by surfaces. We then have formulas for mass and moments, just as for curves or bodies:

Mass: $$M = \iint_S \delta \, d\sigma$$

First moments about coordinate planes:

$$M_{yz} = \iint_S x \delta \, d\sigma, \quad M_{xz} = \iint_S y \delta \, d\sigma,$$

$$M_{xy} = \iint_S z \delta \, d\sigma$$

Coordinates of center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

Moments of inertia about coordinate axes:

$$I_x = \iint_S (y^2+z^2)\,\delta\,d\sigma$$

$$I_y = \iint_S (x^2+z^2)\,\delta\,d\sigma$$

$$I_z = \iint_S (x^2+y^2)\,\delta\,d\sigma$$

$$I_L = \iint_S r^2\,\delta\,d\sigma$$

r = distance from (x,y,z) to line L

ex) Find the center of mass of a conical band

$$z^2 = x^2 + y^2, \qquad 1 \le z \le 3$$

with density $\delta = \frac{1}{z}$.

By symmetry, $\bar{x} = \bar{y} = 0$, so we only need to find $\bar{z}$. We take polar coordinates

$$\vec{c}(r,\theta) \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases} \qquad \begin{matrix} 1 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{matrix}$$

which gives $\vec{c}_r \times \vec{c}_\theta =$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\,\vec{k}$$

and thus $d\sigma = r\sqrt{2}\,dr\,d\theta$.

The mass is then

$$M = \iint_S \delta\, d\sigma = \int_0^{2\pi}\int_1^3 \left(\frac{1}{r}\right)(r\sqrt{2})\,dr\,d\theta$$

$$= 4\pi\sqrt{2}$$

The first moment with respect to the xy-plane is

$$M_{xy} = \iint_S \delta z \, d\sigma$$

$$= \int_0^{2\pi} \int_1^3 \left(\frac{1}{r}\right)(r)\left(r\sqrt{2}\right) dr \, d\theta$$

$$= 2\pi\sqrt{2}\left[\frac{r^2}{2}\right]_1^3 = 8\pi\sqrt{2}$$

$\downarrow$

This gives

$$\bar{z} = \frac{M_{xy}}{M} = \frac{8\pi\sqrt{2}}{4\pi\sqrt{2}}$$

$$= 2$$

So the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 2)$$