§16.7 Stokes' Theorem

For vector fields in the plane,

$$\vec{F} = M(x,y)\,\vec{i} + N(x,y)\,\vec{j},$$

we used the "$\vec{k}$-component of curl"

$$(\nabla \times \vec{F}) \cdot \vec{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

to measure rotation caused by $\vec{F}$.

We will now study the "full" curl operation to describe rotation for vector fields in 3d space.

<u>Def</u> Let

$$\vec{F} = M(x,y,z)\,\vec{i} + N(x,y,z)\,\vec{j} + P(x,y,z)\,\vec{k}$$

be a vector field in space, with M, N, P continuously differentiable. The <u>curl</u> of $\vec{F}$, denoted $\operatorname{curl}\vec{F}$ or $\nabla \times \vec{F}$, is the vector field

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

Note that for vector fields in the xy-plane (so $P = 0$ and $\frac{\partial M}{\partial z} = \frac{\partial N}{\partial z} = 0$), we have

$$\nabla \times (M\vec{i} + N\vec{j}) = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

as before.

Just as in the 2d case, curl $\vec{F}$ has a "paddle-wheel" interpretation. If $\vec{F}$ is the velocity field of a fluid in 3d space, then at a point $Q(x_0, y_0, z_0)$, the fluid will circulate around an axis parallel to $(\text{curl}\,\vec{F})(x_0, y_0, z_0)$. The direction of $(\text{curl}\,\vec{F})(x_0, y_0, z_0)$ is such that the rotation will be counterclockwise when viewed from "above" (i.e. the curl vector is pointing <u>at</u> you). The <u>magnitude</u> of the curl vector describes the <u>velocity</u> of the rotation. We will justify this interpretation in a moment.

<u>ex</u> Find the curl of the vector field

$$\vec{F} = yz\hat{i} - xz\hat{j} + y^2\vec{k}$$

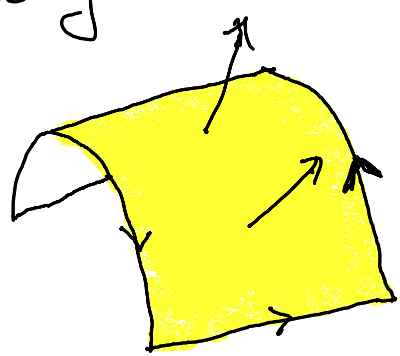
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & -xz & y^2 \end{vmatrix}$$

$$= (2y + x)\vec{i} + y\vec{j} - 2z\vec{k}$$

Stokes' theorem

In two dimensions, Green's theorem related the curl of a vector field over a 2d region with the circulation around its boundary. Stokes' theorem generalizes this to <u>surfaces</u> in 3d space.

First we need to say a word about orientation. Let S be a smooth surface in $\mathbb{R}^3$ with boundary curve C. (We sometimes write $C = \partial S$). Then an orientation $\vec{n}$ of S induces an orientation of C: we orient C such that, when the normal vector $\vec{n}$ is facing you, C is being traversed counterclockwise.



For example, if S is the unit disk in the xy-plane

$$S = \{(x,y,0) : x^2 + y^2 \le 1\}$$

with orientation $\vec{n} = \vec{k}$, then $C = \partial S$ is the unit circle

$$C = \{(x,y,0) : x^2 + y^2 = 1\}$$

with the usual counterclockwise orientation.

With this convention, Stokes' theorem says:

Thm (Stokes)

Let S be a (piecewise) smooth surface with (piecewise) smooth boundary curve C. Let $\vec{F}$ be a vector field whose component functions are continuously differentiable. If $\vec{n}$ is an orientation of S, and we take the induced orientation of C, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma$$

Note that Stokes' theorem implies, in particular, that if two different oriented surfaces $(S_1, \vec{n}_1)$ and $(S_2, \vec{n}_2)$ have the same boundary (i.e. $\partial S_1 = \partial S_2$ and $\vec{n}_1$ and $\vec{n}_2$ induce the same orientation on this common curve), then

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n}_1 \, d\sigma = \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n}_2 \, d\sigma$$

This is analogous to the line integral of a gradient vector field

$$\int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$$

only depending on the endpoints of the curve. So Stoke's theorem is yet another incarnation of the fundamental theorem of calculus.

ex Verify Stokes' theorem for the vector field

$$\vec{F} = 2yz\vec{i} - y^2\vec{j} + xy\vec{k}$$

over the cone

$$S: \; z^2 = x^2 + y^2, \qquad 0 \le z \le 2$$

The curl of $\vec{F}$ is

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2yz & -y^2 & xy \end{vmatrix}$$

$$= x\vec{i} + y\vec{j} - 2z\vec{k}$$

We parametrize S by

$$\vec{c}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r\,\vec{k}$$

$$0 \le r \le 2, \qquad 0 \le \theta \le 2\pi.$$

so

$$\vec{c}_r \times \vec{c}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\,\vec{k}$$

This gives

$$(\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma = (\nabla \times \vec{F}) \cdot (\vec{c}_r \times \vec{c}_\theta)\, dr\, d\theta$$

$$= (r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} - 2r\,\vec{k})$$

$$\cdot (-r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\,\vec{k})\, dr\, d\theta$$

So $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$

$$= \int_0^{2\pi} \int_0^{2} -3r^2 \, dr \, d\theta = \boxed{-16\pi}$$

Now let's do the curve side.

We parametrize the boundary by

$\vec{r}(t) = 2\cos(t) + 2\sin(t) + 2\vec{k}$,

$0 \le t \le 2\pi$

Since $(\vec{C}_r \times \vec{C}_\theta) \cdot \vec{k} = r > 0$, the orientation of the cone is pointing inwards/upwards, so this is the correct orientation for the boundary. Recall our vector field is $\vec{F} = 2yz\,\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$

So

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(8\sin t\,\vec{i} - 4\sin^2 t\,\vec{j} + 4\sin t\,\vec{k}\right)\cdot\left(-2\sin t\,\vec{i} + 2\cos t\,\vec{j}\right) dt$$

$$= -8\int_0^{2\pi} \left(2\sin^2 t + \underbrace{\sin^2 t\cos t}_{=0 \text{ by oddness}}\right) dt$$

$$= -16 \int_0^{2\pi} \sin^2 t \, dt$$

$$= -16 \underbrace{\left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi}}_{\#65 \text{ in Table of Integrals}} = \boxed{-16\pi}$$

Let's also see this for a different surface having the same boundary: the disk D parametrized by

$$\vec{p}(r, \theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + 2\vec{k}$$

$$0 \le r \le 2, \quad 0 \le \theta \le 2\pi$$

with $\vec{p}_r \times \vec{p}_\theta = r\vec{k}$. In this case,

$$\iint_D (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma$$

$$= \int_0^{2\pi}\int_0^{2} \left(r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} - 4\vec{k}\right) \cdot (r\vec{k}) \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (-4r) = -16\pi$$

<u>Paddlewheel interpretation</u>

Now let's justify the interpretation of curl we gave earlier. We can understanding the vector field $\nabla \times \vec{F}$ by considering

$(\nabla \times \vec{F}) \cdot \vec{p}$ for all direction vectors $\vec{p}$. (e.g. just $\vec{p} = \vec{i}, \vec{j}, \vec{k}$ would suffice to know its components)

So consider a fluid with velocity field $\vec{F}$, and let Q be a point in the fluid. We consider a small disk D of radius ρ centered at Q, and contained in a plane normal to $\vec{p}$. This has boundary a circle C of radius ρ. For ρ small,

$(\nabla \times \vec{F})(Q) \cdot \vec{p}$ is approximately equal to the average value of

$(\nabla \times \vec{F}) \cdot \vec{p}$ over D:

$(\nabla \times \vec{F})(Q) \cdot \vec{p}$

$$= \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \iint_D (\nabla \times \vec{F}) \cdot \vec{p} \, d\sigma$$

$$= \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \oint_C \vec{F} \cdot d\vec{r}$$ by Stokes' theorem

The term $\oint_C \vec{F} \cdot d\vec{r}$ is the circulation of $\vec{F}$ around the circle C, and $(\nabla \times \vec{F})(Q) \cdot \vec{p}$ is greatest when $\vec{p}$ is the direction of $(\nabla \times \vec{F})(Q)$. So the circulation is greatest when the normal of our circle is parallel to $\nabla \times \vec{F}$.

To be more precise, consider a fluid rotating counterclockwise around the z-axis with angular velocity ω. This has velocity field

$$\vec{F} = \omega(-y\hat{i} + x\hat{j}).$$

We compute the circulation of $\vec{F}$ around a circle of radius

ρ centered at the origin:

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(-\omega\rho\sin t\,\vec{i} + \omega\rho\cos t\,\vec{j}\right) \cdot \left(-\rho\sin t\,\vec{i} + \rho\cos t\,\vec{j}\right) dt$$

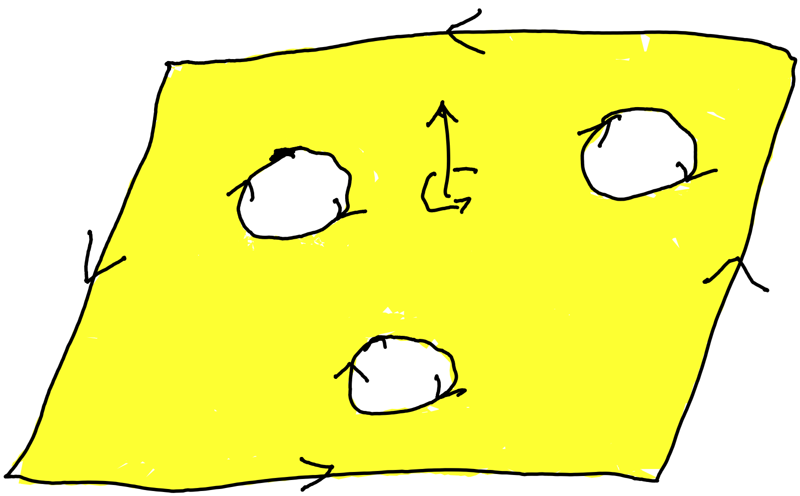
$$= \int_0^{2\pi} \omega\rho^2\,dt = 2\pi\omega\rho^2$$

So the term $\frac{1}{\pi\rho^2}\oint_C \vec{F}\cdot d\vec{r}$ from above becomes 2ω, i.e. twice the angular velocity. In summary: a paddlewheel placed in a fluid with velocity field $\vec{F}$ will rotate counterclockwise around the direction $\dfrac{\nabla \times \vec{F}}{|\nabla \times \vec{F}|}$, with angular velocity $\frac{1}{2}|\nabla \times \vec{F}|$.

<u>Other remarks</u>

Stokes' theorem for surfaces with holes: if a surface has holes, Stokes' theorem still applies, but the boundary is a <u>union</u> of simple closed curves.

Beware that the induced orientation on the inner curves is clockwise!



ex Verify Stokes theorem for

$$\vec{F} = \frac{1}{x^2+y^2}\left(-y\,\vec{i} + x\,\vec{j}\right)$$

over the annulus

$$A: \ 1 \leq x^2 + y^2 \leq 4$$

The curl of $\vec{F}$ is

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = 0$$

so we just need to show that

$$\oint_{\partial A} \vec{F} \cdot d\vec{r} = 0.$$

We write

$\partial A = C_1 \cup C_2$ parametrized by

C_1: $\vec{r}(t) = \cos t\,\vec{i} - \sin t\,\vec{j}$

C_2: $\vec{r}(t) = 2\cos(t)\,\vec{i} + 2\sin(t)\,\vec{j}$

both with $0 \le t \le 2\pi$.

Notice that C_1 is oriented clockwise! Now

$$\oint_{\partial A} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} (\sin t\,\vec{i} + \cos t\,\vec{j}) \cdot (-\sin t\,\vec{i} - \cos t\,\vec{j})$$

$$+ \frac{1}{4}(-2\sin t\,\vec{i} + 2\cos t\,\vec{j}) \cdot (-2\sin t\,\vec{i} + 2\cos t\,\vec{j})\, dt$$

$$= \int_0^{2\pi} 0\, dt = 0,$$ which would not have worked if C_1

were oriented clockwise.

An important identity

The curl of a <u>gradient</u> vector field is zero!

$$\nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial f}{\partial y\,\partial z} - \frac{\partial f}{\partial z\,\partial y}\right)\vec{i} + \left(\frac{\partial f}{\partial z\,\partial x} - \frac{\partial f}{\partial x\,\partial z}\right)\vec{j}$$

$$+ \left(\frac{\partial f}{\partial x\,\partial y} - \frac{\partial f}{\partial y\,\partial x}\right)\vec{k} = 0$$

by symmetry of mixed partials. We write

$$\text{curl}\ \text{grad}\ f = 0 \quad \text{or} \quad \nabla \times \nabla f = 0.$$

More colloquially, we could say that "gradient vector fields are irrotational." (A vector field $\vec{F}$ is called <u>irrotational</u> if $\text{curl}\ \vec{F} = 0$.)

Once again, the converse is true for simply connected domains, but <u>not in general</u>.

For instance, the angular vector field

$$\vec{F} = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$$

is irrotational, but there is no smooth function f with $\vec{F} = \nabla f$. (Morally, $\vec{F}$ is the gradient of

$$\theta(x,y) = \tan^{-1}(y/x),$$

but this is not defined at $x=0$, and not continuous at $y=0$).