1. q-factorials and the norm

Recall from [Ang15] that the Norm map of Witt vectors $W(R) \xrightarrow{N} W(R)$ is given by

$$N(x) = x - V\delta(x)$$

and characterized by

$$FNx = x^p,$$
$$Nx = x \mod V.$$

This descends to a map $W_n(R) \xrightarrow{N} W_{n+1}(R)$ which is multiplicative (and additive mod V), generalizing the Teichmüller lift. The existence of this map is basically the "Frobenius rigidity lemma" (I may be the only one who calls it that)

$$x \equiv y \mod p^n \implies x^p \equiv y^p \mod p^{n+1}.$$

Now let R be a perfectoid ring, and let $A = \mathbf{A}_{\mathrm{inf}}(R)$. How can we understand the Norm at the level of $\mathbf{A}_{\mathrm{inf}}(R)$, rather than $W(R)$? I will view $W_n(R)$ as an A-algebra via the map $\widetilde{\theta}_n\phi^{-1}$, which identifies $W_n(R) = A/[p^n]_A$, where

$$[p^n]_A = \xi\phi(\xi)\cdots\phi^{n-1}(\xi).$$

Let me just do this for $W_1(R) \xrightarrow{N} W_2(R)$, the general case is similar. The above characterization of the Norm becomes: $\mathfrak{N}(x) \in A$ lifts $N(x \bmod \xi) \in W_2(R)$ iff

$$\mathfrak{N}(x) = x^p \bmod \xi$$
$$\mathfrak{N}(x) = \phi(x) \bmod \phi(\xi)$$

Now we can easily write down a general formula for this:

$$\mathfrak{N}(x) = \phi(x) - \frac{\phi(\xi)}{\delta(\xi)}\delta(x).$$

This should look familiar: if $\phi(\xi) \mid \phi(x)$, then Bhatt-Scholze [BS19, Notation 16.1] define the "$[p]_q$-th divided power" of x to be

$$\gamma(x) = \frac{\phi(x)}{\phi(\xi)}\delta(\xi) - \delta(x) \quad !!!$$

Actually, they don't include $\delta(\xi)$, but $\delta(\xi) = 1 \bmod \xi$ over the q-de Rham prism. One hope is that the Norm could be used to develop a version of the q-crystalline site over general prisms.

Note that γ above is really a q-analogue of $\frac{x^p}{p}$, rather than $\frac{x^p}{p!}$; in the p-local context you can get away with this. But we can do better when $R = \mathbf{Z}_p^{\mathrm{cycl}}$, so that $A = \mathbf{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}$ is the perfection of the q-de Rham prism; here $\xi = [p]_{q^{1/p}}$, and more generally $[p^n]_A = [p^n]_{q^{1/p}}$, this is the motivation for the notation.

For $x, y \in A$ with $\delta(x) = \delta(y) = 0$, define the "$q$-twisted power"

$$(x-y)^{[p]_q} = (x-y)(x-qy)\cdots(x-q^{p-1}y)$$

Despite the notation, this depends on x and y, not just on $x - y$. But one checks that $(x-y)^{[p]_q}$ is a lift of $N(x-y)$, distinct from $\mathfrak{N}(x-y)$ unless $p = 2$, since

$$(x-y)^{[p]_q} = (x-y)^p \mod [p]_{q^{1/p}}$$
$$(x-y)^{[p]_q} = \phi(x-y) \mod [p]_q$$

It might seem annoying that γ and $(x-y)^{[p]_q}$ are defined at the level of $\mathbf{A}_{\mathrm{inf}}$, whereas the Norm exists only at the level of the system $W_\bullet(R)$. However, the fact that those expressions *descend* to a map $A/\xi \to A/\xi\phi(\xi)$ is equivalent to the "fundamental lemma of q-crystalline cohomology" [BS19, Lemma 16.7]!

Here is a similar, but more basic phenomenon. In the above normalization, the Frobenius and Verschiebung become

$$\begin{array}{c} W_2(R) \\ {\scriptstyle F}\downarrow \ \uparrow{\scriptstyle V} \\ W_1(R) \end{array} \quad = \quad \begin{array}{c} A/\xi\phi(\xi) \\ {\scriptstyle \mathrm{can}}\downarrow \ \uparrow{\scriptstyle \phi(\xi)/\delta(\xi)} \\ A/\xi \end{array}$$

so the "prism condition" $\phi(\xi) = up \mod \xi$ is equivalent to $FV = p$. This condition is required for the Witt vectors to form what we in equivariant homotopy call a *Mackey functor*; the Norm promotes this to a *Tambara functor*. So these very subtle properties of prisms are equivalent to saying that prisms give rise to an "equivariant commutative ring" in our sense! I think this is very striking.

This is documented in my paper [Sul20, §3.3]. That paper is very difficult, but all the essential ideas are contained in §3.1, which is fairly elementary.

The goal of that paper was to study the *slice filtration*—a variant of the Nygaard filtration which is in some ways more natural from the point of view of equivariant homotopy theory—on perfectoid rings. This should globalize to give a slice filtration on prismatic cohomology. So far I have no idea what this looks like for schemes, but here is what I expect.

Just as the Nygaard filtration $\mathcal{N}^{\geq i}$ is where ϕ is divisible by $[p]_q^i$, the slice filtration $\mathcal{S}^{\geq i}$ should be where ϕ is divisible by $[pi]_q!$ (you can make sense of this over any prism). For instance on the prism itself, we should have

$$\begin{aligned} \mathcal{N}^{\geq i} A &= \xi^i A \\ \mathcal{S}^{\geq i} A &= \xi^i \phi(\xi)^{\lfloor i/p \rfloor} \phi^2(\xi)^{\lfloor i/p^2 \rfloor} \cdots A \end{aligned}$$

Note that these agree for $i < p$, so we expect the slice filtration to mainly be interesting for schemes of dimension $\geq p$. The slice filtration on Δ_X should "stack" scaled copies of the Nygaard filtration of all the Frobenius twists of X, presumably through the Cartier isomorphism. Just as $x \mapsto x^n$ takes $\mathcal{N}^{\geq i}$ to $\mathcal{N}^{\geq in}$, $x \mapsto N(x)$ should take $\mathcal{S}^{\geq i}$ to $\mathcal{S}^{\geq ip}$.

One problem is that I don't know how the Norm works in degrees > 0. One could expect maps

$$W_n \Omega_X^k \xrightarrow{N} W_{n+1} \Omega_X^{kp}$$

and maybe

$$H^k_{\mathrm{cris}}(X/W_n) \xrightarrow{N} H^{kp}_{\mathrm{cris}}(X/W_{n+1})$$

lifting the pth power maps, but I'm not sure yet; translating from the topological story is nontrivial.

The other lead I have is: the formulas in [Sul20, Theorem 1.3], despite looking kind of crazy, are very similar to formulas appearing in work of Gros-Le Stum-Quirós. I'm still trying to understand their work.

2. Prismatic Witt vectors

For an oriented prism (A, ξ), continue to write $[p^n]_A = \xi\phi(\xi)\cdots\phi^{n-1}(\xi)$ as before, and $R = A/\xi$. If A is not perfect, it is no longer true that $A/[p^n]_A \cong W_n(R)$. However, there is still a comparison map

$$W_n(R) \to A/[p^n]_A$$

which is an injection for transversal prisms. This is constructed carefully in [Mol20], but here is the basic idea. For transversal prisms (i.e. R is p-torsionfree), there is an injection

$$A/[p^n]_A \hookrightarrow \prod_{i=0}^{n-1} A/\phi^i(\xi)$$

by [AB19, Lemma 3.7]. Let me call this "transversal coordinates". If we use ghost coordinates on the source and transversal coordinates on the target, then the map is given by

$$\begin{aligned} W_n(R) &\to A/[p^n]_A \\ (w_0, \ldots, w_{n-1}) &\mapsto (w_{n-1}, \phi(w_{n-2}), \ldots, \phi^{n-1}(w_0)) \end{aligned}$$

This is compatible with F in the source (hence the weird reversal) and the projection in the target, so taking the limit, we get a map $\mathbf{A}_{\mathrm{inf}}(R) \to A$. (The isomorphism

$$\varprojlim_F W_n(R) \cong \varprojlim_R W_n(\varprojlim_\varphi R/p)$$

is valid for any p-adic ring, not just perfectoids, so that's what I mean by $\mathbf{A}_{\mathrm{inf}}$). For example, for the Breuil-Kisin prism this is the inclusion $W(k) \hookrightarrow W(k)[\![z]\!] = \mathfrak{S}$.

This formula is easier to understand by considering the case $n = 2$, where we have pullbacks

$$\begin{array}{ccccc} W_2(A/\xi) & \longrightarrow & A/[p^2]_A & \longrightarrow & A/\xi \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{can}} \\ A/\xi & \xrightarrow[\phi]{} & A/\phi(\xi) & \xrightarrow[\mathrm{can}]{} & A/(\xi, \phi(\xi)) \end{array}$$

In the case of the Breuil-Kisin prism, the bottom row is expressing that the Frobenius $\mathcal{O}_K \xrightarrow{\phi} \mathcal{O}_K/p$ extends along $\mathcal{O}_K[\pi^{1/p}]$. So we can think of $A/[p^n]_A$ as "Witt vectors with ramification". (This may be related to ramified Witt vectors, but I don't understand those). More generally, we can write

$$A/[p^n]_A = \{(w_0, \ldots, w_{n-1}) \in W_n(A/\phi^{n-1}(\xi)) \mid w_i \in A/\phi^{n-i-1}(\xi)\}.$$

I would like to characterize the image of $W_n(R)$ in $A/[p^n]_A$ in terms of transversal coordinates. Here is a way to do that (maybe you have to squint a little to see that's what it's doing). Define the *prismatic ghost polynomials*

$$\begin{aligned} w_0^\xi &= a_0 \\ w_1^\xi &= a_0^\phi + \phi(\xi)a_1 \\ w_2^\xi &= a_0^{\phi^2} + \phi^2(\xi)a_1^\phi + \phi([p^2]_A)a_2 \\ w_n^\xi &= \sum_{i=0}^{n} \phi^{n-i+1}([p^i]_A)a_i^{\phi^{n-i}} \end{aligned}$$

Note that unlike the usual ghost polynomials, these are additive (but still not multiplicative). Now define the *prismatic Witt vectors* $W_n^\xi(R)$ by

$$W_n^\xi(R) = \text{image of } (w_0^\xi, \ldots, w_{n-1}^\xi) \text{ in } \prod_{i=1}^{n} A/[p^i]_A.$$

For instance, in the case $n = 2$ we have a pullback

$$\begin{array}{ccc} W_2^\xi(R) & \longrightarrow & A/\xi \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \phi} \\ A/[p^2]_A & \xrightarrow[\text{can}]{} & A/\phi(\xi) \end{array}$$

Proposition/Conjecture: $W_n^\xi(R) \cong W_n(R)$. The Proposition part is that this is almost immediate from the topological perspective: the definition is rigged so that $W_n^\xi(R) \cong \mathrm{TR}_0^n(R/\mathbf{S}_A)$, while we know that $\mathrm{TR}_0^n(R) = W_n(R)$. But the absolute and relative TR should agree on π_0. Here $\mathbf{S}_A$ (which isn't guaranteed to exist) is such that $\mathbf{S}_A \otimes_{\mathbf{S}} \mathbf{Z} = A$; for example in the Breuil-Kisin case $\mathbf{S}_{\mathfrak{S}} = \mathbf{S}_{W(k)}[\![z]\!]$, where $\mathbf{S}_{W(k)}$ is Lurie's "spherical Witt vectors". The Conjecture part is to prove this purely algebraically; it's essentially a version of the Cartier-Dieudonné-Dwork lemma.

The "A-analogues" might also give a way to calculate prismatic cohomology in local coordinates, generalizing the q-de Rham complex. Set $[n]_A = [p^{v_p(n)}]_A$, so that $\phi([n]_A) = un \mod \xi$ for a unit u (depending on n). This is ugly, but will work up to units. Then in local coordinates we can define

$$\nabla_A(x^n) = [n]_A x^n \operatorname{dlog} x$$

to get a version of $A\Omega$. This is lacking the more elegant formula $\nabla_q f(x) = \frac{f(qx)-f(x)}{qx-x}$ that we have for the q-derivative, but otherwise behaves the same.

3. Floating rings

Consider the ring $A = \mathbf{Z}[q]$. This has two interesting pieces of structure: Adams operations $\psi^n(q) = q^n$ (lifting Frobenius when n is prime), and q-analogues $[n]_q = \frac{q^n-1}{q-1}$. This should be the basic example of an "integral prism", whatever that is.[1]

So let's see how to encode this. For any semiring B, write $B^\bullet$ for the underlying multiplicative monoid of B. Of course the Adams operations give a monoid map

$$\mathbf{N}^\bullet \to \mathrm{End}_{\mathrm{Ring}}(A).$$

q-analogues are *semi-multiplicative* with respect to this action:

$$[mn]_q = \frac{q^{mn}-1}{q^n-1}\frac{q^n-1}{q-1} = \psi^n([m]_q)[n]_q$$

That is, Adams operations and q-analogues assemble to give a map

$$\begin{gathered} \mathbf{N}^\bullet \to A^\bullet \rtimes \mathrm{End}_{\mathrm{Ring}}(A) \\ n \mapsto ([n]_q, \psi^n) \end{gathered}$$

This is cool because now we can put any ring on the right-hand side and any monoid on the left-hand side.

[1] One might complain that we should complete at $(q-1)$, but Mao [Zho21] has recently given a decompleted notion of prism: rather than $\delta(\xi) \in A^\times$, one asks that $\delta(\xi) \in (A/\xi)^\times$.

Example: a map $C_2 \to A^\bullet \rtimes \mathrm{End}_{\mathrm{Ring}}(A)$ consists of an involution $A \to A$, $z \mapsto \bar{z}$ with $\bar{\bar{z}} = z$, together with an element $\epsilon \in A$ such that $\epsilon\bar{\epsilon} = 1$ ("of norm one", if you like). This is precisely the input for Hermitian K-theory! (Taking $\epsilon = 1$ corresponds to symmetric bilinear forms, $\epsilon = -1$ corresponds to skew-symmetric bilinear forms.) Actually this gives the wrong thing for non-commutative rings, but you can fix that by taking into account C_2 action on Ring that sends $A \mapsto A^{\mathrm{op}}$.

The second observation is that the right-hand side is actually the endomorphisms of A in a new category. For any category $\mathcal{C}$ equipped with a functor $\mathcal{C} \xrightarrow{U} \mathrm{Mon}$ to the category of monoids, we can define a new category $\mathcal{D}$ which has the same objects as $\mathcal{C}$, but where

$$\mathcal{D}(X, Y) = U(Y) \times \mathcal{C}(X, Y)$$
$$\mathrm{End}_{\mathcal{D}}(X) = U(X) \rtimes \mathrm{End}_{\mathcal{C}}(X)$$

The fancy way to say this is we compose with the delooping functor $\mathrm{Mon} \xrightarrow{B} \mathrm{Cat}$ and apply the Grothendieck construction.

The motivating example of this is the functor $\mathrm{Vect}_k \to \mathrm{Mon}$ sending a k-vector space V to the monoid $(V, +)$. Then this construction gives the category of *affine spaces* over k! An affine space is the same thing as a vector space, but there are more morphisms between them. So we think of the functor U as specifying "translations" that we want to add into our category.

So apply this construction to the functor $\mathrm{Ring} \to \mathrm{Mon}$ sending $A \mapsto A^\bullet$. It would be horrible to say "affine ring", so let me call this the category of "floating rings". A floating ring is the same thing as a ring, but a map of floating rings (or a "floating map" of rings) is a ring homomorphism times a constant.

Then we can interpret the structure on $\mathbf{Z}[q]$, the input for Hermitian K-theory, and p-typical (oriented) prisms as representations of $\mathbf{N}^\bullet$, C_2, and $(\mathbf{N}, +) \cong p^{\mathbf{N}} \subset \mathbf{N}^\bullet$ respectively in the category of floating rings.

The category of floating rings is equivalent to the category of pairs (A, M) where $A \in \mathrm{Ring}$ and M is a free A-module of rank one, since the category of such M is a model of $BA^\bullet$. This makes it look a little less exotic. Anyway, this seems like it could be a very interesting category.

Along with the observations in §1, this strongly suggests that there should be a generalization of "prism" associated to a monoid M acting on a compact Lie group G. This would presumably be related to Dress-Siebeneicher's G-Witt vectors, which I don't understand.

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