THE ENERGY OF GRAPHS AND MATRICES

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Abstract. Let $G$ be a finite, undirected, and simple graph. If $\{v_1, \cdots, v_n\}$ is the set of vertices of $G$, then the adjacency matrix $A(G) = [a_{ij}]$ is an $n$-by-$n$ matrix where $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and $a_{ij} = 0$ otherwise. The energy of a graph, $E(G)$, is defined as the sum of the absolute values of eigenvalues of $A(G)$. The concept of energy originates in chemistry and was first defined by I. Gutman in 1978. It has been generalized recently as follows: For a graph $G$ on $n$ vertices, let $M$ be a matrix associated with $G$. Let $\mu_1, \cdots, \mu_n$ be the eigenvalues of $M$ and let $\bar{\mu}$ be the average of $\mu_1, \cdots, \mu_n$. The more general $M$-energy of $G$ is then defined as:

$$E_M(G) = \sum_{i=1}^{n} |\mu_i - \bar{\mu}|.$$ 

In this paper we present our results on graph energy when $M$ is the Laplacian matrix, the signless Laplacian matrix, or the distance matrix. In particular we give bounds for energy of different graph classes and study the effect of edge deletion.

1. Introduction

Throughout this paper $G$ will denote a simple (no loops or multiple edges), undirected graph with $n$ vertices and $m$ edges. If $\{v_1, \cdots, v_n\}$ is the set of vertices of $G$, then the adjacency matrix of $G$, $A(G) = A = [a_{ij}]$ is an $n$-by-$n$ matrix, where $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and $a_{ij} = 0$ otherwise. Thus, $A$ is a symmetric $(0,1)$-matrix with real eigenvalues and zeros on the diagonal. If $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $A$, then we have

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \lambda_1 + \lambda_2 + \cdots + \lambda_n = 0. \tag{1}$$

The energy of $G$ was first defined by Gutman in 1978 as the sum of the absolute values of the eigenvalues of $A(G)$ [1]:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$
If \( G \) is not connected, then \( E(G) \) is the sum of the energy of its connected components. However, we do not make the assumption that \( G \) is connected. From (1) we obtain

\[
\sum \{ \lambda_i : \lambda_i > 0 \} = -\sum \{ \lambda_i : \lambda_i < 0 \} = \frac{E(G)}{2}.
\]

The concept of energy originated in chemistry. Hückel molecular orbital theory is a field of theoretical chemistry where graph eigenvalues occur. The carbon atoms of a hydrocarbon system correspond to vertices of a graph associated with the molecule. From Hückel theory, the energy of a molecular graph is equal to the total \( \pi \)-electron energy of a conjugated hydrocarbon [2].

The concept of energy has been generalized in two different directions. Let \( A \) be an \( m \)-by-\( n \) matrix and \( A^* \) denote its adjoint (conjugate transpose of \( A \)). The singular values \( s_1(A) \geq s_2(A) \geq \cdots \geq s_m(A) \) of a matrix \( A \) are the square roots of the eigenvalues of \( AA^* \). Note that if \( A \) is an \( n \)-by-\( n \) Hermitian matrix (i.e., \( A = A^* \)), then the singular values of \( A \) are the absolute values of the eigenvalues of \( A \).

For any \( A \in \mathcal{M}_{m,n} \) define the energy of \( A \), \( \mathcal{E}(A) \),

\[
\mathcal{E}(A) = \sum_{i=1}^{m} s_i(A).
\]

From above, we note that the usual energy of a graph \( G \), \( E(G) = \mathcal{E}(A(G)) \).

Another generalization of energy is defined as follows: let \( M \) be a matrix associated with \( G \). Suppose \( \mu_1, \cdots, \mu_n \) are the eigenvalues of \( M \) and \( \bar{\mu} \) is the average of \( \mu_1, \cdots, \mu_n \). The \( M \)-energy of \( G \) is then defined as the absolute deviation

\[
E_M(G) = \sum_{i=1}^{n} |\mu_i - \bar{\mu}|.
\]

If \( M \) is the adjacency matrix \( A(G) \), then \( \bar{\mu} = 0 \) using (1). Hence, the usual energy \( E(G) = E_A(G) \).

In this paper we consider Laplacian energy, signless Laplacian energy, Distance energy and Incidence energy. The different energies are defined in section 2. Our results on Laplacian and signless Laplacian are given in section 3. In section 4, we present our results on Distance energy and Incidence energy. In the Appendix section, we include an example of integral calculation for Laplacian and a catalog of eigenvalues for reference.

2. Background

2.1. Graph Theory Preliminaries. A complete graph on \( n \) vertices will be denoted by \( K_n \). Given a graph \( G \), a subgraph \( H \) is said to be induced if it contains all edges in \( G \) joining two vertices of \( H \). Let \( G - H \) denote a subgraph of \( G \) obtained by removing all vertices \( H \). We denote by \( G - E(H) \) the subgraph of \( G \) obtained by deleting all edges of \( H \) but keeping all vertices of \( H \).
2.2. Laplacian Energy.

**Definition 2.1.** The *degree matrix* of $G$, $D(G)$, is an $n$-by-$n$ diagonal matrix such that $D(G)_{ii} = d_i$ where $d_i$ is the degree of vertex $v_i$ (the number of edges incident with $v_i$).

**Definition 2.2.** The *Laplacian matrix* of a graph, $L(G)$, is defined as

$$L(G) = D(G) - A(G),$$

where $D(G)$ is the degree matrix, and $A(G)$ is the adjacency matrix of $G$.

The Laplacian matrix is both a positive semidefinite and an $M$-matrix [3]. Since the sum of the degrees of vertices of a graph $G$ is $2m$, we note that the trace of $L(G)$ is $2m$. Therefore, the mean of its eigenvalues is $\frac{2m}{n}$.

**Definition 2.3.** Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of $L(G)$. Then the *Laplacian energy*, $LE(G)$, is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$  

2.3. Signless Laplacian Energy.

**Definition 2.4.** The *signless Laplacian matrix* of a graph, $L^+(G)$, is defined as

$$L^+(G) = D(G) + A(G)$$

where $D(G)$ is the degree matrix, and $A(G)$ is the adjacency matrix of $G$. Thus, the entries of the signless Laplacian matrix equal the absolute values of those in the Laplacian matrix.

**Definition 2.5.** Let $\nu_1, \ldots, \nu_n$ be the eigenvalues of $L^+(G)$. Then the *signless Laplacian energy*, $LE^+(G)$, is defined as

$$LE^+(G) = \sum_{i=1}^{n} \left| \nu_i - \frac{2m}{n} \right|.$$  

2.4. Distance Energy.

**Definition 2.6.** The *distance matrix*, $D(G)$, is the matrix $D(G) = [d_{ij}]$ where $d_{ij}$ is the length of the shortest path between $v_i$ and $v_j$. If $v_i$ is an isolated vertex, then, for any $j$, the entry $d_{ij}$ is taken to be 0 instead of $\infty$.

Since we take $G$ to be a simple graph, no vertex is adjacent to itself. Therefore, the entries on the diagonal of $D(G)$ are all 0, and the mean of the eigenvalues is 0.

**Definition 2.7.** The *distance energy* of $G$, $DE(G)$, is defined as

$$DE(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_i$ are the eigenvalues of the distance matrix.
2.5. Incidence Energy.

**Definition 2.8.** The *incidence matrix* $I(G)$, is an $n$-by-$m$ matrix for which

$$I(G)_{i,j} = \begin{cases} 
1 & \text{if } v_i \text{ is incident with the edge } e_j \\
0 & \text{otherwise}
\end{cases}$$

Notice that the incidence matrix in general is not a square matrix.

**Definition 2.9.** The *incidence energy* of $G$, $IE(G)$, is defined as

$$IE(G) = \sum_{i=1}^{n} s_i(I(G))$$

where $s_i$ are the singular values of $I(G)$.

3. Results on Laplacian and Signless Laplacian Energy

In this section we present some results on Laplacian and signless Laplacian energy. The first result gives upper and lower bounds for the Laplacian energy of a graph when some of its edges are removed. The proof of our theorem on edge deletion is an application of Ky Fan’s inequality for singular values [4]. We begin by stating the Ky Fan inequality in Theorem 3.1.

**Theorem 3.1 ([4]).** Let $X$, $Y$, and $Z$ be $n$-by-$n$ complex matrices such that $X+Y=Z$. Then

$$\sum_{i=1}^{n} s_i(X) + \sum_{i=1}^{n} s_i(Y) \geq \sum_{i=1}^{n} s_i(Z)$$

where $s_i$ are the singular values of the indicated matrix.

**Theorem 3.2.** Let $H$ be an induced subgraph of a simple graph $G$. Suppose $\tilde{H}$ denotes the union of $H$ and vertices of $G - H$ (as isolated vertices). Then

$$LE(G) - LE(\tilde{H}) \leq LE(G - E(H)) \leq LE(G) + LE(\tilde{H})$$

where $E(H)$ is the edge set of $H$.

**Proof.** Let $D$ be the degree matrix of $G$. It is easy to observe that

$$D(G) = D(\tilde{H}) + D(G - E(H)).$$

Since

$$A(G) = \begin{bmatrix} A(H) & X^T \\
X & A(G - H) \end{bmatrix}$$

where $X$ corresponds to the edges connecting $H$ and $G - H$. We can write

$$A(G) = \begin{bmatrix} A(H) & 0 \\
0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & X^T \\
X & A(G - H) \end{bmatrix} = A(\tilde{H}) + A(G - E(H)).$$
Therefore,
\[ L(G) = D(G) - A(G) = L(\tilde{H}) + L(G - E(H)). \]

Since \( m = |E(\tilde{H})| + |E(G - E(H))| \), we observe that
\[ L(G) - \frac{2m}{n} I = \left( L(\tilde{H}) - \frac{2|E(\tilde{H})|}{n} I \right) + \left( L(G - E(H)) - \frac{2|E(G - E(H))|}{n} I \right). \]

Hence, by applying Ky Fan’s inequality, we have
\[ \text{(2)} \quad LE(G) \leq LE(\tilde{H}) + LE(G - E(H)). \]

On the other hand,
\[ L(G - E(H)) = L(G) + (-L(\tilde{H})) \]
and
\[ L(G - E(H)) - \frac{2|E(G - E(H))|}{n} I = \left( L(G) - \frac{2m}{n} I \right) + \left[ - \left( L(\tilde{H}) - \frac{2|E(\tilde{H})|}{n} I \right) \right]. \]

Since
\[ \sum_i s_i \left[ - \left( L(\tilde{H}) - \frac{2|E(\tilde{H})|}{n} I \right) \right] = LE(\tilde{H}) \]
we have, by Ky Fan’s inequality,
\[ \text{(3)} \quad LE(G - E(H)) \leq LE(G) + LE(\tilde{H}). \]

From (2) and (3), we get
\[ LE(G) - LE(\tilde{H}) \leq LE(G - E(H)) \leq LE(G) + LE(\tilde{H}). \]

\[ \square \]

The following theorem for signless Laplacian energy has a proof very similar to Theorem 3.2. Therefore, we have stated Theorem 3.3 and omitted its proof.

**Theorem 3.3.** Let \( H \) be an induced subgraph of a simple graph \( G \). Suppose \( \tilde{H} \) denotes the union of \( H \) and vertices of \( G - H \) (as isolated vertices). Then
\[ LE^+(G) - LE^+(\tilde{H}) \leq LE^+(G - E(H)) \leq LE^+(G) + LE^+(\tilde{H}) \]

**Lemma 3.4.** Suppose \( \tilde{H} \) consists of \( K_2 \) and \( n - 2 \) isolated vertices. Then
\[ LE(\tilde{H}) = \frac{4(n - 1)}{n}. \]

**Proof.** Clearly,
\[ L(\tilde{H}) = \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}. \]
Then
\[
\det[\lambda I - L(\tilde{H})] = \det \begin{bmatrix}
\lambda - 1 & 1 & 0 & 0 \\
1 & \lambda - 1 & 0 & 0 \\
0 & 0 & \lambda & \cdots & 0 \\
& & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda
\end{bmatrix}
\]
\[= ([\lambda - 1]^2 - 1)\lambda^{n-2}
= \lambda^{n-2}(\lambda^2 - 2\lambda)
= \lambda^{n-1}(\lambda - 2).
\]
Therefore eigenvalues of \(L(\tilde{H})\) are 2, with multiplicity 1, and 0, with multiplicity \((n - 1)\).
Thus,
\[
LE(\tilde{H}) = \left(2 - \frac{2}{n}\right) + (n - 1)\left(\frac{2}{n}\right) = 2 + (n - 2)\left(\frac{2}{n}\right)
= \frac{2n + 2n - 4}{n} = \frac{4(n - 1)}{n}.
\]

**Corollary 3.5.** Suppose \(H\) is a single edge \(\{e\}\) and \(\tilde{H}\) consists of \(\{e\}\) and \(n - 2\) isolated vertices. Then
\[
LE(G) - \frac{4(n - 1)}{n} \leq LE(G - \{e\}) \leq LE(G) + \frac{4(n - 1)}{n}.
\]

**Lemma 3.6.** Let \(\alpha \in \mathbb{R}\). Then \(E(L(G(u,v)) - \alpha I) = (n - 1)|\alpha| + |2 - \alpha|\), where
\[
E(C) = \sum_{i=1}^{n} |\lambda_i(C)|.
\]

**Proof.** We have
\[
L(G(u,v)) - \alpha I = \begin{bmatrix}
1 - \alpha & -1 & 0 & \cdots & 0 \\
-1 & 1 - \alpha & \vdots & \ddots & \\
0 & \cdots & -\alpha & \cdots & 0 \\
& & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & -\alpha
\end{bmatrix}
= \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
Where \(A\) is a \(2 - b\) by \(2 - b\) matrix with \(1 - \alpha\) on the diagonal and \(-1\) on the off diagonal, \(B\) and \(C\) are zero matrices, and \(D\) is a matrix with \(-\alpha\) along the diagonal and zeros everywhere else.

**Case 1:** \(\alpha \neq 0 \Rightarrow A\) is nonsingular.
Since \(\det(L(G(u,v)) - \alpha I) = \det(A)\det(D - CA^{-1}B) = (-2)^{n-2}(\alpha - 2)(\alpha)\)\(n-1\), the singular values of \(L(G(u,v)) - \alpha I\) are \(|\alpha - 2|\), with multiplicity 1, and \(|\alpha|\) with multiplicity \(n - 1\), which proves the lemma.

**Case 2:** \(\alpha = 0\)
Since \( \alpha = 0 \), \( \mathcal{E}(L(G(u,v)) - 0I_n) = \mathcal{E}(L(G(u,v))) \). From the lemma of Result 9 (ask Duy when putting in the report) we have that the eigenvalues of \( L(G(u,v)) \) are 0 with multiplicity \( n - 1 \) and 2 with multiplicity 1. Therefore, \( \mathcal{E}(L(G(u,v))) = 2 \), which follows Lemma 3.6 when \( \alpha = 0 \).

\[ \text{Theorem 3.7.} \]

\[ LE(G) \leq LE(\tilde{H}) + LE(G - E(H)) \leq 4m(1 - \frac{1}{n}) \]

with equalities on both sides holding \( \iff \) \( E(H) = 0 \) or \( G \) is the union of one edge and \( (n - 2) \) isolated vertices.

\[ \text{Proof.} \] From Theorem 3.2 we have that \( LE(G) - LE(\tilde{H}) \leq LE(G - E(H)) \iff LE(G) \leq LE(\tilde{H}) + LE(G - E(H)) \). Thus the left inequality in (4) follows.

We now prove the right inequality. Using an expression of \( L(G) \) by Spielman, we have:

\[ L(G) = \sum_{(u,v) \in E(G)} L(G(u,v)) \]

where

\[ L(G(u,v))_{i,j} = \begin{cases} 1 & \text{if } (i,j) = (u,u) \text{ or } (i,j) = (v,v) \\ 1 & \text{if } (i,j) = (u,v) \text{ or } (i,j) = (v,u) \\ 0 & \text{otherwise.} \end{cases} \]

Let \( E = \{e_1, e_2, \ldots, e_m\} \), where \( e_i = \{u_i, v_i\} \) and \( i = 1, \ldots, n \). Then \( L(G) - \frac{2m}{n} I_n = \sum_{i=1}^{m} (L(G(e_i)) - \frac{2}{n} I_n) \). Let

\[ X_i = L(G(e_i)) - \frac{2m}{n} I_n. \]

So by (5),

\[ L(G) - \frac{2m}{n} I_n = \sum_{(u,v) \in E(G)} (L(G(u,v)) - \frac{2}{n} I_n) \]

\[ = \sum_{i=1}^{m} (L(G(e_i)) - \frac{2}{n} I_n) = \sum_{i=1}^{m} X_i. \]

On the other hand,

\[ L(G) = L(\tilde{H}) + L(G - E(H)) \]
\[
\L(G) - \frac{2m}{n} I_n = L(\tilde{H}) + L(G - E(H)) - \frac{2m}{n} I_n
\]

\[
\L(G) - \frac{2m}{n} I_n = \sum_{e_i \in E(H)} L(G(e_i)) + \sum_{e_i \in (G - E(H))} L(G(e_i)) - \frac{2m}{n} I_n
\]

\[
\sum_{e_i \in E(H)} (L(G(e_i)) - \frac{2}{n} I_n) + \sum_{e_i \in (G - E(H))} (L(G(e_i)) - \frac{2}{n} I_n)
\]

\[
\sum_{e_i \in E(H)} X_i + \sum_{e_i \in (G - E(H))} X_i.
\]

Note that

\[
\sum_{e_i \in E(H)} X_i + \sum_{e_i \in (G - E(H))} X_i = \left[ L(\tilde{H}) - \frac{2E(\tilde{H})}{n} I_n \right] + \left[ L(G - E(H)) - \frac{2E(G - E(H))}{n} I_n \right].
\]

So, applying the Ky Fan inequality, we have

\[
\varepsilon(L(\tilde{H}) - \frac{2E(\tilde{H})}{n} I_n) + \varepsilon(L(G - E(H)) - \frac{2E(G - E(H))}{n} I_n)
\]

\[
= \varepsilon(\sum_{e_i \in E(H)} X_i) + \varepsilon(\sum_{e_i \in (G - E(H))} X_i) \leq \sum_{i=1}^{m} \varepsilon(X_i).
\]

Thus, by Lemma 3.6,

\[
LE(\tilde{H}) + LE(G - E(H)) \leq 4m(1 - \frac{1}{n}),
\]

which proves the right inequality of (4). The relation (4) follows from combining the left and right inequalities.

**Equality cases:** From [Robbiano, Jimenez], we know that equality for \(LE(G) \leq 4m(1 - \frac{1}{n})\) holds if and only if \(E(G) = \emptyset\) or if \(G\) is the union of one edge and \(n - 2\) isolated vertices. From (4),

\[
LE(G) \leq LE(\tilde{H}) + LE(G - E(H)) \leq 4m(1 - \frac{1}{n}).
\]

We make the observation that equality in \(LE(G) \leq 4m(1 - \frac{1}{n})\) also implies the equalities in (4), i.e.

\[
LE(G) = LE(\tilde{H}) + LE(G - E(H)) = 4m(1 - \frac{1}{n}).
\]

\[\square\]

**Corollary 3.8.** Given the relation

\[
LE^+(G) \leq LE^+(\tilde{H}) + LE^+(G - E(H)) \leq 4m(1 - \frac{1}{n}),
\]

equality holds on both sides if and only if the graph \(G\) has no edges or \(G\) is the union of one edge and \(n - 2\) isolated vertices.
Proof. We can define the signless Laplacian matrix of $G$ as:

$$L^+(G(u,v))_{i,j} = \begin{cases} 
1 & \text{if } (i,j) = (u,u) \text{ or } (i,j) = (v,v) \\
1 & \text{if } (i,j) = (u,v) \text{ or } (i,j) = (v,u) \\
0 & \text{otherwise}
\end{cases}$$

Note that $L^+(G(u,v) - \alpha I) = (n - 1)|\alpha| + |2 - \alpha|$, by following the proof in Lemma 3.6. Following the same steps as for Theorem 3.7, we prove the corollary.

Nordhaus and Gaddum [9] gave bounds for the sum of the chromatic numbers of a graph and its complement. The next theorem gives a Nordhaus-Gaddum type relationship for Laplacian energy. The proof uses the Ky Fan inequality and the following bound for Laplacian energy from [5].

**Theorem 3.9** ([5]). For a graph $G$ with $n$ vertices and $m$ edges,

$$LE(G) \leq 2M,$$

where $M = m + \frac{1}{2} \sum_{i=1}^{n} (d_i - \frac{2m}{n})^2$ and $d_i$ is the degree of vertex $v_i$.

**Theorem 3.10.** For a graph $G$ and its complement $\bar{G}$,

$$2(n - 1) \leq LE(G) + LE(\bar{G}) \leq n(n - 1).$$

**Proof.** We first prove the left inequality.

We observe that

$$L(G) + L(\bar{G}) = nI - J$$

where $I$ is the identity matrix and $J$ is an $n$-by-$n$ matrix with all entries equal to 1. Then

$$L(G) - \frac{2m}{n}I + L(\bar{G}) - \frac{2}{n} \left( \frac{n(n-1)}{2} - m \right) I = nI - J - \frac{2m}{n}I - \frac{2}{n} \left( \frac{n(n-1)}{2} - m \right) I$$

$$= I - J.$$

Thus, by the Ky Fan inequality,

$$LE(G) + LE(\bar{G}) \geq \sum_{i=1}^{n} s_i(I - J)$$

where $s_i$ are the singular values of $I - J$.

By Sylvester’s Determinant theorem,

$$det(A + cd^T) = det(A)(1 + d^T A^{-1}c).$$

We apply this to $I - J$ where

$$A = \begin{pmatrix} 
\lambda - 1 & 0 & \ldots & 0 \\
0 & \lambda - 1 & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & \lambda - 1 \\
\end{pmatrix} \quad \text{and} \quad c = d = \begin{pmatrix} 
1 \\
\vdots \\
1 \\
\end{pmatrix}.$$
Therefore, the eigenvalues of $I - J$ are 1, with multiplicity $(n - 1)$, and $(n - 1)$ with multiplicity 1.

Thus,

$$(n - 1) + (n - 1) = 2(n - 1) \leq LE(G) + LE(\bar{G}).$$

The right inequality is proved as follows:

Applying Theorem 3.9, we find that

$$LE(G) + LE(\bar{G}) \leq 2(M + \bar{M})$$

where

$$\bar{M} = \bar{m} + \frac{1}{2} \sum_{i=1}^{n} \left( \bar{d}_i - \frac{2\bar{m}}{n} \right)^2$$

$$= \frac{n(n - 1)}{2} - m + \frac{1}{2} \sum_{i=1}^{n} \left( d_i - \frac{2m}{n} \right)^2.$$ 

We calculate that

$$M + \bar{M} = \frac{n(n - 1)}{2} - \frac{4m^2}{n} + \sum_{i=1}^{n} d_i^2.$$ 

Thus,

$$LE(G) + LE(\bar{G}) \leq n(n - 1) - \frac{8m^2}{n} + 2 \sum_{i=1}^{n} d_i^2.$$ 

Now using the fact that $\sum_{i=1}^{n} d_i^2 \leq m(m + 1)$ [11], we see that

$$LE(G) + LE(\bar{G}) \leq n(n - 1) - \frac{8m^2}{n} + 2m(m + 1).$$ 

Since $G$ is a graph of $n$ vertices, the right side of the inequality can be written as $\frac{f(m)}{n}$, where

$$f(m) = m^2(2n - 8) + 2mn + n^2(n - 1).$$ 

When we fix $n > 4$, we observe that $f(m) > 0$ for any value of $m \geq 0$. Since the graph of $f(m)$ is a parabola opening upwards, the minimum value of $f(m)$ occurs at $m = 0$. Therefore,

$$LE(G) + LE(\bar{G}) \leq \frac{n^2(n - 1)}{n} = n(n - 1),$$

thus proving the right inequality. For $n \leq 4$, the inequality has been verified using the catalog in Appendix 1.

The following is an integral formula for the energy of a graph by Coulson [10] (see also [12]).

**Theorem 3.11.** If $G$ is a graph on $n$ vertices, then

$$E(G) = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \left[ n - \frac{ix\phi'(ix)}{\phi(ix)} \right] dx.$$ 

where $\phi$ is the characteristic polynomial of $A(G)$ and $p.v.$ represents the principal value of the integral.
In the following theorem we extend the integral formula to the Laplacian energy of $G$.

**Theorem 3.12.** If $G$ is a graph on $n$ vertices and $m$ edges, then

$$LE(G) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \left[ n - \frac{ix\phi_L(ix)}{\phi_L(ix)} \right] dx,$$

where $\phi_L$ is the characteristic polynomial of $L(G) - \frac{2m}{n} I$.

**Proof.** Denote by $\gamma_1, \gamma_2, \cdots, \gamma_p$ the distinct eigenvalues of $L(G) - \frac{2m}{n} I$, and by $n_j$ the algebraic multiplicity of $\gamma_j$. Notice $n_1 + n_2 + \cdots + n_p = n$.

Since $L(G) - \frac{2m}{n} I$ is a real, symmetric matrix, we may assume $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_p$. Then

$$\phi_L(z) = \prod_{j=1}^{p} (z - \gamma_j)^{n_j}$$

and we have

$$\frac{\phi_L'(z)}{\phi_L(z)} = \sum_{j=1}^{p} \frac{n_j}{z - \gamma_j}.$$

Define $f(z) = z \frac{\phi_L'(z)}{\phi_L(z)} = \sum_{j=1}^{p} \frac{zn_j}{z - \gamma_j}$. Then $f$ is an analytic function except for simple poles at $\gamma_1, \cdots, \gamma_p$. Note that $f$ has no pole at $z = 0$.

Let $\Gamma$ be a contour that goes along the $y$-axis from $(0, r)$ to the point $(0, -r)$ and then returns to $(0, r)$ along a semicircle of radius $r > \gamma_1$.

![Diagram](image)

By Cauchy’s residue theorem

$$\oint_{\Gamma} f(z) dz = 2\pi i \sum_{\gamma_j > 0} \text{Res}(\gamma_j).$$

Since $\gamma_j$ is a simple pole, $\text{Res}(f, \gamma_j) = \lim_{z \to \gamma_j} (z - \gamma_j)f(z) = \gamma_j n_j$. Since the trace of $L(G) - \frac{2m}{n} I$ is zero, we get

$$\frac{1}{\pi i} \oint_{\Gamma} f(z) dz = 2 \sum_{\gamma_j > 0} \gamma_j n_j = LE(G).$$
For \( n \) being a constant, using Cauchy’s theorem,
\[
\frac{1}{2\pi i} \oint_{\Gamma} f(z)dz = \frac{1}{2\pi i} \oint_{\Gamma} [f(z) - n]dz.
\]

Since
\[
f(z) - n = \frac{z\phi'_L(z)}{\phi_L(z)} - n
\]
\[
= \sum_{j=1}^{p} \frac{zn_j}{z - \gamma_j} - \sum_{j=1}^{p} n_j
\]
\[
= \sum_{j=1}^{p} \left( \frac{zn_j}{z - \gamma_j} - n_j \right)
\]
\[
= \sum_{j=1}^{p} \frac{n_j \gamma_j}{z - \gamma_j},
\]
we have
\[
|f(z) - n| \leq \sum_{j=1}^{p} \left| \frac{n_j \gamma_j}{z - \gamma_j} \right| \leq \sum_{j=1}^{p} \frac{n_j |\gamma_j|}{|z| - |\gamma_j|}.
\]

On \(|z| = r\), we get
\[
|f(z) - n| \leq \sum_{j=1}^{p} \frac{n_j |\gamma_j|}{r - |\gamma_j|}.
\]

Taking the limit of \( r \to \infty \) we get \( |f(z) - n| \to 0 \) on \( C_r^+ \), which is the right half of the circle of radius \( r \). Now
\[
\oint_{\Gamma} [f(z) - n]dz = \int_{-r}^{r} [f(iy) - n]d(iy) + \int_{C_r^+} [f(z) - n]dz.
\]

When \( r \to \infty \),
\[
\frac{1}{\pi i} \oint_{\Gamma} [f(z) - n]dz = \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} i[f(iy) - n]dy
\]
\[
= \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} [n - f(iy)]dy.
\]

Therefore,
\[
LE(G) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \left[ n - \frac{ix\phi'_L(ix)}{\phi_L(ix)} \right] dx
\]
\[
= \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \left[ n - x \frac{d}{dx} \log \phi_L(ix) \right] dx.
\]

\[\square\]

**Remark 3.13.** A similar argument shows the following:

1. Suppose \( \phi_{L+} \) denotes the characteristic polynomial of \( L^+(G) - \frac{2m}{n} I \). Then
\[
LE^+(G) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \left[ n - x \frac{d}{dx} \log \phi_L(ix) \right] dx.
\]
Suppose $\phi_D$ denotes the characteristic polynomial of $D(G)$. Then

$$DE(G) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \left[ n - x \frac{d}{dx} \log \phi_D(ix) \right] dx.$$ 

**Corollary 3.14.** If $G$ is a simple graph with $n$ vertices, then

$$LE(G) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{dx}{x^2} \log \left[ x^n \phi_L \left( \frac{i}{x} \right) \right]$$

where $\phi_L$ is the characteristic polynomial of $L(G) - \frac{2m}{n} I$.

**Proof.** From Theorem 3.12,

$$LE(G) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \left[ n - \frac{ix \phi_L' (ix)}{\phi_L (ix)} \right] \left( \frac{-1}{t^2} \right) dt + \frac{1}{\pi} \text{p.v.} \int_{0}^{\infty} \left[ n - \frac{ix \phi_L' (ix)}{\phi_L (ix)} \right] \left( \frac{-1}{t^2} \right) dt$$

Substituting $x = \frac{1}{t}$, it follows that

$$LE(G) = \frac{1}{\pi} \text{p.v.} \int_{0}^{\infty} \left[ n - \frac{t \phi_L' (i \frac{1}{t})}{\phi_L \left( \frac{1}{t} \right)} \right] \left( \frac{-1}{t^2} \right) dt + \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{0} \left[ n - \frac{t \phi_L' (i \frac{1}{t})}{\phi_L \left( \frac{1}{t} \right)} \right] \left( \frac{-1}{t^2} \right) dt$$

Let $u = \frac{1}{t}$ and

$$dv = \left[ n - \frac{t \phi_L' (i \frac{1}{t})}{\phi_L \left( \frac{1}{t} \right)} \right] dt.$$

Note that $du = -\frac{dt}{t^2}$ and $v = \log \left[ t^n \phi_L \left( \frac{i}{t} \right) \right] v$.

Using integration by parts, we have

$$LE(G) = \frac{1}{\pi} \lim_{r \to -\infty} \left( \frac{1}{t} \log \left[ t^n \phi_L \left( \frac{i}{t} \right) \right] \right)_{-r}^{+r} + \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{1}{t^2} \log \left[ t^n \phi_L \left( \frac{i}{t} \right) \right] dt.$$

Using L'Hospital’s rule

$$\lim_{t \to \infty} \left( \frac{\log \left[ t^n \phi_L \left( \frac{i}{t} \right) \right]}{t} \right) = \lim_{t \to \infty} \left[ \frac{nt^{n-1} \phi_G \left( \frac{1}{t} \right) + t^n \phi_G' \left( \frac{1}{t} \right)}{t^n \phi_G \left( \frac{1}{t} \right)} \right]_{-t}^{+t} = \frac{n}{t} + \frac{\phi_G' \left( \frac{1}{t} \right)}{\phi_G \left( \frac{1}{t} \right)} \left( \frac{-i}{t^2} \right) = 0,$$

because $\phi_G \left( \frac{1}{t} \right)$ is finite when $t \to \infty$.

Thus,

$$\lim_{r \to -\infty} \left( \frac{1}{t} \log \left[ t^n \phi_L \left( \frac{i}{t} \right) \right] \right)_{-r}^{+r} = 0.$$
Hence, we conclude that

\[ LE(G) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{1}{t^2} \log \left[ t^n \phi_L \left( \frac{i}{t} \right) \right] dt. \]

\[ \square \]

4. Results on Distance and Incidence Energy

This section provides results on distance and incidence energy. We begin by stating some known inequalities for singular values in Theorem 4.1 and Corollary 4.2. Using these facts we go on to prove a result on distance energy change due to deletion of edges in a cut set. We also prove a relationship between a graph and an induced subgraph.

**Theorem 4.1** ([6]). For a partitioned matrix \( C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix} \) where both \( A \) and \( B \) are square matrices, we have

\[ \sum_j s_j(A) + \sum_j s_j(B) \leq \sum_j s_j(C). \]

**Corollary 4.2** ([6]). For a partitioned matrix \( C = \begin{bmatrix} A & X \\ Y & B \end{bmatrix} \) where \( A \) and \( B \) are square matrices, we have

\[ \sum_j s_j(A) \leq \sum_j s_j(C). \]

Equality holds if and only if \( X, Y, \) and \( B \) are all zero matrices.

**Theorem 4.3.** Let \( H \) be an induced subgraph of a simple graph \( G \). Then \( DE(H) \leq DE(G) \) and equality holds if and only if \( E(H) = E(G) \), where \( E(G) \) represents the edges in \( G \).

**Proof.** Consider the distance matrix

\[ D(G) = \begin{bmatrix} D(H) & X^T \\ X & D(G - H) \end{bmatrix} \]

where the matrix \( X \) represents the edges that connect \( G - H \) and \( H \). By Corollary 4.2, \( DE(H) \leq DE(G) \).

For the equality case, suppose \( E(H) = E(G) \). Then

\[ D(G) = \begin{bmatrix} D(H) & 0 \\ 0 & 0 \end{bmatrix}. \]

Thus, \( DE(G) = DE(H) \).

Now suppose that the equality holds. Then

\[ \sum_j s_j(D(G)) = \sum_j s_j(D(H)). \]

Therefore, \( D(G - H) \) and \( X \) are zero matrices. Thus \( E(G) = E(H) \). \( \square \)
Corollary 4.4. For any simple graph $G$ with at least 1 edge, $DE(G) \geq 2$.

Definition 4.5. [6] If $E$ is a set of edges of $G$ such that $G - E$ is the union of two complementary induced subgraphs, then $E$ is called a cut set of $G$ [6].

Theorem 4.6. If $E$ is a cut set of a simple graph $G$, then $DE(G - E) \leq DE(G)$.

Proof. Since $E$ is a cut set of $G$, $G - E = H \bigoplus K$ where $H$ and $K$ are two complementary induced subgraphs of $G$. Apply Theorem 4.1 to

$$D(G) = \begin{bmatrix} D(H) & X^T \\ X & D(K) \end{bmatrix}. $$

Then

$$\sum_i s_i(D(H)) + \sum_i s_i(D(K)) \leq \sum_i s_i(D(G)).$$

Since $G - E$ is the union of the complementary induced subgraphs $H$ and $K$, we get

$$DE(G - E) \leq DE(G).$$

\[ \square \]

Theorem 4.7 gives bounds for the distance energy of a graph $G$. In our following results, we use these bounds to establish bounds and a relationship for the distance energies of path and star graphs.

Theorem 4.7 ([13]). For a graph on $n$ vertices,

$$\sqrt{\frac{2}{n-1}} \sum_{1 \leq i < j \leq n} (d_{ij})^2 \leq DE(G) \leq \sqrt{\frac{2n}{n-1}} \sum_{1 \leq i < j \leq n} (d_{ij})^2.$$ 

Theorem 4.8. For a path on $n$ vertices,

$$\sqrt{\frac{n^2(n-1)(n+1)}{6}} \leq DE(P_n) \leq \sqrt{\frac{n^3(n-1)(n+1)}{6}}.$$ 

Proof. For a path on $n$ vertices, we have

$$\sum_{1 \leq i < j \leq n} (d_{ij})^2 = (n-1)(1^2) + (n-2)(2^2) + \cdots + (2)(n-2)^2 + (1)(n-1)^2$$

$$= \sum_{i=1}^{n-1} (n-i)i^2$$

$$= \frac{n^2(n-1)(n+1)}{12}.$$ 

Therefore, using Theorem 4.7,

$$\sqrt{\frac{n^2(n-1)(n+1)}{6}} \leq DE(P_n) \leq \sqrt{\frac{n^3(n-1)(n+1)}{6}}.$$ 

\[ \square \]

Theorem 4.9. The eigenvalues of the distance matrix of a star graph on $n$ vertices, $S_n$ (with $n \geq 3$), are the following: $-2$ with multiplicity $n - 2$, $n - 2 - \sqrt{n^2 - 3n + 3}$ with multiplicity 1 and $n - 2 + \sqrt{n^2 - 3n + 3}$ with multiplicity 1.
Proof. The characteristic polynomial of $S_n$ is the determinant of the following matrix:

$$D(S_n) - \lambda I = \begin{bmatrix}
-\lambda & 1 & 1 & \ldots & 1 \\
1 & -\lambda & 2 & \ldots & 2 \\
1 & 2 & -\lambda & \ldots & 2 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & 2 & \ldots & -\lambda \\
\end{bmatrix}.$$ 

We partition the matrix as

$$D(S_n) - \lambda I = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$ 

where

$$A = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2 & \ldots & 2 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 2 & \ldots & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} -\lambda & 2 & \ldots & 2 \\ 2 & -\lambda & \ldots & 2 \\ \vdots & \ddots & \ddots & \vdots \\ 2 & \ldots & -\lambda \end{bmatrix}.$$ 

We prove by contradiction that $A$ is always invertible. Suppose $A$ is not invertible. Then $D(S_n)$ has an eigenvalue of 1 or -1.

If $\lambda = 1$ is an eigenvalue, there must exist a nonzero vector such that

$$\begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 2 & \ldots & 2 \\ 1 & 2 & 0 & \ldots & 2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 2 & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. $$

Solving the resulting system, we obtain

$$x_1 = x_2 = \ldots = x_n = 0,$$

which contradicts the assumption of a nonzero vector. Therefore, $\lambda = 1$ cannot be an eigenvalue of $D(S_n)$.

Similarly, if $\lambda = -1$ is an eigenvalue, there must exist a nonzero vector such that

$$\begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 2 & \ldots & 2 \\ 1 & 2 & 0 & \ldots & 2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 2 & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (-1) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

which can be reduced to the previous situation. Hence, $\lambda = -1$ cannot be an eigenvalue of $D(S_n)$ and $A$ is always invertible.

If the matrix $A$ is nonsingular, then $\lambda$ is not 1 or -1 and the following relation holds:

$$\det(D(S_n) - \lambda I) = \det(A) \cdot \det(D - CA^{-1}B)$$
It results that

$$det(D - CA^{-1}B) = \begin{vmatrix} -\lambda^3 + 6\lambda + 4 & 2\lambda^2 + 5\lambda + 2 & \cdots & 2\lambda^2 + 5\lambda + 2 \\ \frac{2\lambda^2 + 5\lambda + 2}{\lambda^2} & -\lambda^3 + 6\lambda + 4 & \cdots & \frac{2\lambda^2 + 5\lambda + 2}{\lambda^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2\lambda^2 + 5\lambda + 2}{\lambda^2} & \cdots & -\lambda^3 + 6\lambda + 4 & \frac{2\lambda^2 + 5\lambda + 2}{\lambda^2} \end{vmatrix}. $$

Let $\alpha = -\frac{\lambda^3 + 6\lambda + 4}{\lambda^2}$ and let $\beta = -\frac{\lambda^3 + 6\lambda + 4}{\lambda^2}$.

Then, applying Sylvester’s determinant theorem,

$$det(D - CA^{-1}B) = (\alpha - \beta)^{n-2} + \frac{\beta \cdot (n-2)}{\alpha - \beta}. $$

So the determinant of $D(S_n) - \lambda I$ is

$$det(D(S_n) - \lambda I) = det(A) \cdot det(D - CA^{-1}B)
= (\lambda^2 - 1) \cdot \frac{(\lambda^3 - 2\lambda^2 + \lambda + 2)^{n-2}}{\lambda^2 - 1} \cdot \frac{-\lambda^3 - 2\lambda^2 + \lambda + 4 + (n-2)(2\lambda^2 + 5\lambda + 4)}{-\lambda^3 - 2\lambda^2 + \lambda + 2}.
$$

Hence, the roots of the polynomial are: $-2$ with multiplicity $n-2$, $n-2 - \sqrt{n^2 - 3n + 3}$ with multiplicity 1 and $n-2 + \sqrt{n^2 - 3n + 3}$ with multiplicity 1.

Theorem 4.10. For $n \geq 2$,

$$DE(S_n) \leq DE(P_n).$$

Proof. The result follows directly from Theorem 4.8 and Theorem 4.9.

In Theorem 4.13, we prove Nordhaus-Gaddum type relationship for incidence energy. We need the following lemma and result on a bound for incidence energy [7].

Lemma 4.11. Let $I$ be the identity matrix and $J$ the matrix with all entries equal to 1. Then, the eigenvalues of $(n-2)I + J$ are $n-2$, with multiplicity $n-1$, and $2(n-1)$, with multiplicity 1. Furthermore,

$$\sum_{i=1}^{n} \sqrt{\mu_i((n-2)I + J)} = (n-1)\sqrt{n-2} + \sqrt{2(n-1)}.$$

where $\mu_1, \mu_2, \cdots, \mu_n$ are the eigenvalues of $(n-2)I + J$.

Proof. We can express $\mu I - [(n-2)I + J] = A + cd^T$ where $A$ is the diagonal matrix with entries equal to $\mu - n + 2$. Furthermore, $c$ and $d$ are the column vectors where $c$ has entries equal to 1, and entries of $d$ equal to $-1$.

Case 1: $det(A) \neq 0 \iff \mu \neq n - 2$. Using Sylvester’s determinant theorem, we have

$$det(\mu I - [(n-2)I + J]) = det(A + cd^T) = det(A)det(1 + d^TA^{-1}c) = 0.$$

Since $det(A) \neq 0$, it follows that

$$det(1 + d^TA^{-1}c) = 0 \Rightarrow \mu - 2n + 2 = 0 \Rightarrow \mu = 2(n-1), \text{ multiplicity}=1.$$
Case 2: \( \det(A) = 0 \iff \mu = n - 2 \)

Thus, \( \mu I - [(n - 2)I + J] = -J \Rightarrow \det(\mu I - [(n - 2)I + J]) = 0 \Rightarrow \mu = n - 2 \) is an eigenvalue of \((n - 2)I + J\). Furthermore, from case 1, since the multiplicity of \( \mu = 2(n - 1) \) is 1, the multiplicity of \( \mu = n - 2 \) is \( n - 1 \). The sum of the square roots of \( \mu_i \) follow by straightforward calculation. \( \square \)

**Theorem 4.12** ([7]). For a graph \( G \) with \( m \) edges,
\[
IE(G) \leq \sqrt{2m}.
\]

**Theorem 4.13.** Let \( e_1, \ldots, e_m \) be the edges of \( G \) and \( e_{m+1}, \ldots, e_{\frac{n(n-1)}{2}} \) be the edges of \( \bar{G} \). Then
\[
(n-1)\sqrt{n-2} + \sqrt{2(n-1)} \leq IE(G) + IE(\bar{G}) \leq \frac{n(n-1)}{\sqrt{2}}.
\]

**Proof.** Let \( \hat{I}(G) \) denote the extended incidence matrix of \( G \). We define \( \hat{I}(G) \) as an \( n \)-by-\( \frac{n(n-1)}{2} \) matrix such that
\[
\hat{I}(G)_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ is incident with the edge } e_j \\ 0 & \text{otherwise.} \end{cases}
\]

Observe that \( \hat{I}(G) \) can be partitioned into \( I(G) \) and a 0-matrix. Thus,
\[
\hat{I}(G)\hat{I}(G)^T = I(G)I(G)^T.
\]

It follows that \( I(G) \) and \( \hat{I}(G) \) have the same nonzero singular values. Furthermore, we have \( \hat{I}(G) + \hat{I}(\bar{G}) = I(K_n) \) and
\[
(I(K_n))^2 = (n-2)I + J.
\]

Applying Ky Fan’s inequality, we obtain
\[
\sum_{i=1}^{n} s_i(\hat{I}(G)) + \sum_{i=1}^{n} s_i(\hat{I}(\bar{G})) \geq \sum_{i=1}^{n} s_i(I(K_n))
\]
\[
\iff \sum_{i=1}^{n} s_i(I(G)) + \sum_{i=1}^{n} s_i(I(\bar{G})) \geq \sum_{i=1}^{n} \sqrt{\mu_i((n-2)I + J)}.
\]

Hence, by Lemma 4.11,
\[
IE(G) + IE(\bar{G}) \geq (n-1)\sqrt{n-2} + \sqrt{2(n-1)}.
\]

By Theorem 4.12, we have \( IE(G) \leq \sqrt{2m} \). Similarly, for \( \bar{G} \), we have
\[
IE(\bar{G}) \leq \sqrt{2}\left(\frac{n(n-1)}{2} - m\right).
\]

Therefore,
\[
IE(G) + IE(\bar{G}) \leq \sqrt{2}\left(\frac{n(n-1)}{2}\right) = \frac{n(n-1)}{\sqrt{2}}.
\]

\( \square \)
If $G$ and $H$ are two graphs, let $u \in V(G)$ and $v \in V(H)$. We define the coalescence of $G$ and $H$, denoted $G \circ H$, to be the graph obtained by identifying the vertices $u$ and $v$.

**Theorem 4.14.** For any two graphs $G$ and $H$, $DE(G \circ H) \leq DE(G) + DE(H)$.

**Proof.** From [8], it is known that

$$D(G \circ H) = \begin{pmatrix} D(G - u) & X & 0 \\ X^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Y^T \\ 0 & Y & D(H - v) \end{pmatrix}.$$ 

Let

$$A = \begin{pmatrix} D(G - u) & X & 0 \\ X^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Y^T \\ 0 & Y & D(H - v) \end{pmatrix}.$$ 

Then

$$DE(G \circ H) = \sum_i s_i(D(G \circ H)).$$

$$DE(G) = \sum_i s_i(A).$$

$$DE(H) = \sum_i s_i(B).$$

We apply Ky Fan’s inequality, which yields that $\sum_i s_i(D(G \circ H)) \leq \sum_i s_i(A) + \sum_i s_i(B)$. Thus,

$$DE(G \circ H) \leq DE(G) + DE(H).$$

Equality holds if and only if $X$ or $Y$ is a zero vector (i.e. $u$ or $v$ is an isolated vertex). It is straightforward to check that, when $X$ or $Y$ is a zero vector, equality in the Theorem holds.

If the equality in the Theorem holds, then there exists an orthogonal matrix $P$ such that $PA$ and $PB$ are positive semi-definite. Now let $P$ be partitioned according to the matrix $D(G \circ H)$:

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}.$$ 

Then

$$P^T = \begin{pmatrix} P_{11}^T & P_{12}^T & P_{13}^T \\ P_{21}^T & P_{22}^T & P_{23}^T \\ P_{31}^T & P_{32}^T & P_{33}^T \end{pmatrix}.$$ 

Since $P$ is an orthogonal matrix, $P^TP = I$. Thus,

$$P_{11}^TP_{11} + P_{21}^TP_{21} + P_{31}^TP_{31} = I;$$

(6)
(7) \[ P_{13}^T P_{13} + P_{23}^T P_{23} + P_{33}^T P_{33} = I, \]

(8) \[ P_{11}^T P_{13} + P_{21}^T P_{23} + P_{31}^T P_{33} = 0. \]

Furthermore, we have that
\[
PA = \begin{pmatrix}
P_{11}R + P_{12}X^T & P_{11}X & 0 \\
P_{21}R + P_{22}X^T & P_{21}X & 0 \\
P_{31}R + P_{32}X^T & P_{31}X & 0
\end{pmatrix},
\]

where \( R = D(G - u) \), and
\[
PB = \begin{pmatrix}
0 & P_{13}Y + P_{13}S \\
0 & P_{23}Y + P_{23}S \\
0 & P_{33}Y + P_{33}S
\end{pmatrix},
\]

where \( S = D(H - v) \), are positive semi-definite. So, by symmetry, \( P_{31}X = 0 \) and \( P_{13}Y = 0 \).

Now multiply (8) by \( X^T \) from the left and by \( Y \) from the right:
\[
X^T P_{11} P_{13} Y + X^T P_{21} P_{23} Y + X^T P_{31} P_{33} Y = (P_{21}X)^T (P_{23}Y) = 0.
\]

Notice that \( (P_{21}X)^T \) and \( P_{23}Y \in \mathbb{R} \), so one of these two scalars must be 0.

**Case 1.** \( P_{21}X = 0 \).

Note that \( P_{21}X \) is a diagonal entry of a positive semi-definite matrix \( PA \), but \( P_{21}X = 0 \), so \( P_{11}X = 0 \). By [6], if \( A = a_{ij} \) is a positive semi-definite matrix and \( a_{ii} = 0 \) for some \( i \), then \( a_{ji} = a_{ij} = 0 \) for all \( j \).

Now multiply (6) by \( X^T \) on the left and by \( X \) on the right. Thus,
\[
X^T P_{11}^T P_{11}X + X^T P_{21}^T P_{21}X + X^T P_{31}^T P_{31}X = X^T X = 0.
\]

(i.e. \( X = 0 \) implies the \( u \) is an isolated vertex of \( G \).)

**Case 2.** \( P_{23}Y = 0 \).

Note \( P_{23}Y \) is a diagonal entry of the positive semi-definite matrix \( PB \) but \( P_{23}Y = 0 \), so \( P_{13}Y = 0 \). From (7),
\[
Y^T Y = Y^T P_{13}^T P_{13}Y + Y^T P_{23}^T P_{23}Y + Y^T P_{33}^T P_{33}Y = 0.
\]

(i.e. \( Y = 0 \) implies that \( v \) is an isolated vertex of \( H \).)


5. Appendix I

**Example 5.1.** An Integral Calculation for $LE(C_3)$

We have the characteristic polynomial of $L(C_3) - \frac{2m}{n}$, where $m = 3$ and $n = 3$, is

$$\phi(z) = (z + 2)(z - 1)^2.$$ 

Also,

$$n - \frac{ix\phi'(ix)}{\phi(ix)} = 3 - ix\left(\frac{2}{ix - 1} + \frac{1}{ix + 2}\right)$$

$$= \frac{-6}{(ix - 1)(ix + 2)}.$$ 

By integral formula for Laplacian energy of $C_3$, we have:

$$LE(C_3) = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{-6}{(ix - 1)(ix + 2)} \, dx$$

$$= \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{-6}{i(x - \frac{1}{i})(x + \frac{3}{i})} \, dx$$

$$= \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{6}{(x + i)(x - 2i)} \, dx$$

$$= \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{6(x - i)(x + 2i)}{(x + i)(x - i)(x - 2i)(x + 2i)} \, dx$$

$$= \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{6x^2 + ix + 2}{(x^2 + 1)(x^2 + 4)} \, dx$$

Using partial fractions, we have

$$\frac{6(x^2 + ix + 2)}{(x^2 + 1)(x^2 + 4)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 4},$$

where $A = 2i$, $C = -2i$, $B = 2$, and $D = 4$. Hence,

$$LE(C_3) = \frac{1}{\pi} \left[ p.v. \int_{-\infty}^{+\infty} \frac{2ix + 2}{x^2 + 1} \, dx + p.v. \int_{-\infty}^{+\infty} \frac{-2ix + 4}{x^2 + 4} \, dx \right]$$

$$= \frac{1}{\pi} \left[ i.p.v. \int_{-\infty}^{+\infty} \frac{2x}{x^2 + 1} \, dx + 2p.v. \int_{-\infty}^{+\infty} \frac{dx}{x^2 + 4} - i.p.v. \int_{-\infty}^{+\infty} \frac{2x}{x^2 + 4} \, dx + 2p.v. \int_{-\infty}^{+\infty} \frac{\frac{1}{2} \, dx}{(\frac{x}{2})^2 + 1} \right]$$

$$= \frac{1}{\pi} lim_{r \to -\infty} \left[ ilog(x^2 + 1) \bigg|_{-r}^{+r} - ilog(x^2 + 4) \bigg|_{-r}^{+r} + 2tan^{-1}(x) \bigg|_{-r}^{+r} + 2tan^{-1}(\frac{x}{2}) \bigg|_{-r}^{+r} \right]$$

$$= \frac{1}{\pi} lim_{r \to -\infty} \left[ ilog\left(\frac{x^2 + 1}{x^2 + 4}\right) \bigg|_{-r}^{+r} + 2\pi + 2\pi \right]$$

$$= 4.$$
6. Appendix II

We have calculated the energy, Laplacian energy, signless Laplacian energy, and distance energy for all graphs \( G \) with \(|V(G)| \leq 6\). The graph names correspond to the catalog found in [14].

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7. Acknowledgments

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References