Computer-assisted proofs in PDEs: the dispersive case

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- Simple ideas, qualitative information of the solution.
- Computer-assisted proof.
- Applicable to other “bad” situations: low regularity problems, even in unstable / ill-posed regimes
- Special functions are your friend, not the enemy

The Whitham Equation

Consider the KdV equation:

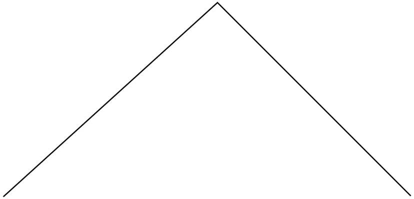
$$v_t - 6vv_x + v_{xxx} = 0$$

"Issues":

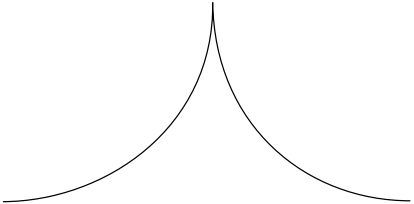
- Local.
- Does not capture many phenomena: wave breaking, sharp crests, non-smooth solutions, etc.

Interested in (singular) solutions of greatest height:

- Corners:



- Cusps:



KdV features a 2nd order approximation of the full dispersion relation of gravity water waves on finite depth:

$$\left(\frac{\tanh(\xi)}{\xi}\right)^{\frac{1}{2}} \sim 1 - \frac{1}{6}\xi^2$$

"Better approximation": change the linear part in KdV using the full dispersion relation.

The Whitham equation

Whitham proposed

$$\partial_t v + 2vv_x + Lv_x = 0,$$

$$\widehat{Lv}(\xi) = \left(\frac{\tanh(\xi)}{\xi}\right)^{\frac{1}{2}} \hat{v}(\xi)$$

as a shallow water approximation.

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For small ξ:

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$\Rightarrow$ Whitham $\sim$ KdV for small frequencies and small times, different for large times.

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For large ξ:

$$\left(\frac{\tanh(\xi)}{\xi}\right)^{\frac{1}{2}} \sim \frac{1}{\xi^{\frac{1}{2}}}$$

$\Rightarrow$ Whitham is a very weakly dispersive perturbation of Burgers.

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Whitham's conjecture

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There exists a limiting traveling wave of $C^{\frac{1}{2}}$ *regularity.*

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Conjecture (Ehrnström-Wahlén, 2016)

Whitham's highest wave is everywhere convex and its asymptotic behavior at 0 *is*

$$v(x,t) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}}|x - \mu t|^{1/2} + o(|x - \mu t|)\,.$$

Here, μ *is part of the problem and needs to be found.*

Water waves & Whitham

Water waves	Whitham
Existence Amick-Fraenkel-Toland, Plotnikov-Toland, 80's.	Existence Ehnrström-Wahlén, 2016
Convexity Plotnikov-Toland, 2004	Convexity Enciso-JGS-Vergara, 2018
Local uniqueness Fraenkel, 2007	???
Uniqueness Kobayashi, 2010	???

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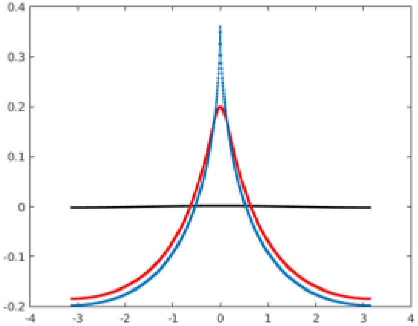
Theorem (Enciso–JGS–Vergara, 2018)

There exists a 2π-periodic highest cusped traveling wave of the Whitham equation which is a convex, $C^{1/2}$ function and behaves asymptotically as

$$v(x,t) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}}|x - \mu t|^{1/2} + O(|x - \mu t|^{1+\eta})$$

for some $\eta > 0$.



The limiting wave is at the end of the branch.

Proof

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- The Whitham equation becomes

$$L\varphi - \mu\varphi + \varphi^2 = 0\,, \quad \widehat{L\varphi} = \left(\frac{\tanh(\xi)}{\xi}\right)^{\frac{1}{2}} \hat{\varphi}(\xi)\,.$$

- Whitham's heuristic argument: crest cusped with $\varphi(x) \sim \frac{\mu}{2} - c|x|^{1/2}$.

Proof

Imposing $u(x) = \frac{\mu}{2} - \varphi(x - \mu t)$ and through the symmetries of the equation we can get rid of μ. In particular, $u(x)$ satisfies the reduced equation:

$$u^2 = \mathcal{L}u,$$

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$$\mathcal{L}u = \int_{-\pi}^{\pi} \big(K(x-y) + K(x+y) - 2K(y)\big)u(y)dy$$

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and K is related to the dispersive multiplier.
Once u is known we can recover μ via

$$\mu\left(1 - \frac{\mu}{2}\right) = 4 \int_0^{\pi} K(y)u(y).$$

Step 0: We reduced the problem to only find u.

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- Write $u = u_0 + \bar{u}$, where $\bar{u}$ is expected to be very small: $O(\varepsilon)$. Then:

$$2u_0\bar{u} - \mathcal{L}\bar{u} = -\bar{u}^2 - \left(u_0^2 - \mathcal{L}u_0\right)$$
$$(I - \frac{1}{2u_0}\mathcal{L})\bar{u} = \frac{1}{2u_0}\left(-\bar{u}^2 - \left(u_0^2 - \mathcal{L}u_0\right)\right)$$

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If we can invert $(I - \frac{1}{2u_0}\mathcal{L})$:

- First term of RHS: $O(\varepsilon^2)$
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Close using a fixed point argument $\Rightarrow$ Explicit estimates of $\|\bar{u}\|$ (small).

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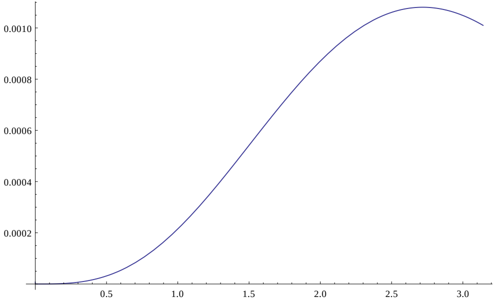
Expected: u_0 strictly convex $\Rightarrow$ $u_0 + \bar{u}$ strictly convex

Tasks

1. Construct a good u_0.
2. Prove that $(I - \frac{1}{2u_0}\mathcal{L})$ is invertible.
3. Check that the involved (explicit) constants are “small enough”.

Step 1: Construction of a good approximation

- Formal asymptotics: very good at $x \ll 1$, terrible at $x \gg 1$. Nontrivial exponents: $u_0 \sim c_1\sqrt{x} + c_2 x^{1.11120\dots} + \dots$



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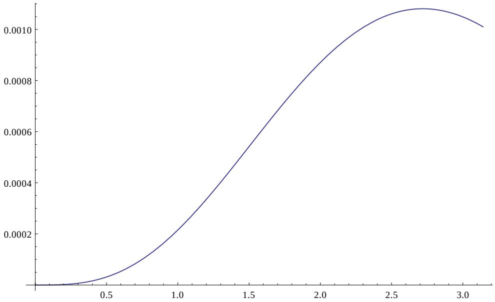

- Formal asymptotics + correction: we add $\sum_{n=1}^{N} b_n(\cos(nx) - 1)$ for some N, b_n. Better global control. In our case $N = 11$.

- Special functions save the day: (Clausen functions)

$$C_z(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^z}, \quad S_z(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^z}.$$

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- Approximate solution u_0= combination of Clausen functions + trigonometric polynomials:

$$u_0(x) = \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk} \Big(\zeta\big(3/2 + kp_0 + jp_1\big) - C_{\frac{3}{2}+kp_0+jp_1}(x)\Big)$$
$$+ \sum_{n=1}^{N_2} b_n \big(\cos(nx) - 1\big),$$

  where a_{jk} b_k are real, p_j solve the equation

$$\frac{\Gamma(-1/2 - p_j)}{\Gamma(-1 - p_j)} \big(1 - \cot(\tfrac{\pi}{2} p_j)\big) = \frac{2}{\sqrt{\pi}},$$

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- We choose the above coefficients so that the defect is small when measured in L^∞:

$$u_0(x) = \sum_{k=0}^{N_j} \sum_{j=0}^{1} A_{jk} |x|^{\frac{1}{2}+kp_0+jp_1} + O(|x|^2)$$

$$\mathcal{L}u_0(x) = \sum_{k=0}^{N_j} \sum_{j=0}^{1} \tilde{A}_{jk} |x|^{1+kp_0+jp_1} + O(|x|^2)\,,$$

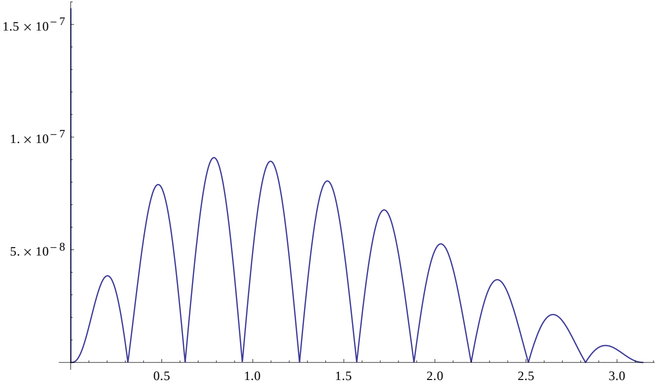
with A_{jk} and $\tilde{A}_{jk}$ real (combinations of the previous a_{jk}).

- Nonlinear system of equations for the coefficients $A_{jk}\,, \tilde{A}_{jk}$:

$$u_0^2(x) - \mathcal{L}u_0(x) = O(|x|^p)\,,$$

for a sufficiently large power p.

Error measured in $L^\infty((0,\pi])$:



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- Instead, use monotonicity (in x) of $C_z(x)$ and

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- Multiprecision ($\sim$ 100 bits) needed.

Step 2: The linear part is invertible

Obs: $\frac{1}{2u_0(x)}\mathcal{L}$ is compact, but it doesn't help to bootstrap.
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$$\left\| \frac{1}{2u_0(x)}\mathcal{L} \right\|_\infty = \sup_x \int |K_0(x,y)|dy\,.$$
$$= 0.99736\ldots$$

$\Rightarrow (I - \frac{1}{2u_0}\mathcal{L})$ is invertible and $\left\|(I - \frac{1}{2u_0}\mathcal{L})^{-1}\right\|_\infty \leq \frac{1}{1-0.99736\ldots} \sim 380\,.$

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- To compute $\int |K_0(x,y)|dy$ for small x we exploit the asymptotics

$$K(x) = \frac{1}{\sqrt{2\pi|x|}} + K_{\mathrm{reg}}(x)\,,$$

with K_{reg} real analytic.

Rigorous Singular integrals

$$
\begin{aligned}
Hf(0) &= -\frac{PV}{\pi} \int \frac{f(y)}{y} dy \\
&= \frac{PV}{\pi} \int_{|y|<\varepsilon} \frac{f(0) - f(y)}{y} dy - \frac{PV}{\pi} \int_{|y|>\varepsilon} \frac{f(y)}{y} dy
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$$\frac{PV}{\pi} \int_{|y|<\varepsilon} \frac{f(0)-f(y)}{y} dy \in \frac{PV}{\pi} \int_{|y|<\varepsilon} -f'([-\varepsilon, \varepsilon]) dy \in -\frac{2\varepsilon f'([-\varepsilon, \varepsilon])}{\pi}$$

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$$\frac{PV}{\pi}\int_{|y|<\varepsilon} \frac{f(0)-f(y)}{y}dy \in \frac{PV}{\pi}\int_{|y|<\varepsilon} -f'([-\varepsilon,\varepsilon])dy \in -\frac{2\varepsilon f'([-\varepsilon,\varepsilon])}{\pi}$$

Obs: The boundary $|y| < \varepsilon$ can be optimized.

Fixed point argument

- For convenience, we write: $u = u_0 + |x| v_0$, where v_0 satisfies

$$(I - T_0)v_0 = \frac{1}{2|x|u_0}\left((\mathcal{L}u_0 - u_0^2) - |x|^2 v_0^2\right).$$

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- For small enough error $\mathcal{L}u_0 - u_0^2$,

$$v_0 \mapsto (1 - T_0)^{-1}\left(\frac{1}{2|x|u_0}\left((\mathcal{L}u_0 - u_0^2) - |x|^2 v_0^2\right)\right)$$

maps a ball of radius $\epsilon_0 \ll 1$ in L^∞ into itself and is contractive.

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$$(I - T_0)v_0 = \frac{1}{2|x|u_0}\left((\mathcal{L}u_0 - u_0^2) - |x|^2 v_0^2\right).$$

- For small enough error $\mathcal{L}u_0 - u_0^2$,

$$v_0 \mapsto (1 - T_0)^{-1}\left(\frac{1}{2|x|u_0}\left((\mathcal{L}u_0 - u_0^2) - |x|^2 v_0^2\right)\right)$$

  maps a ball of radius $\epsilon_0 \ll 1$ in L^∞ into itself and is contractive.

- This only proves the existence of a solution with almost the conjectured asymptotic behavior. $\Rightarrow$ Perturbation of the weight.

Completion of the proof

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- Very sensitive numbers (too delicate estimates / too small numbers to be done by hand)
- We are not using too much special structure of the equation.

New results

Theorem (Dahne–JGS, forthcoming)

There exists a 2π-periodic highest cusped traveling wave of the Burgers-Hilbert equation

$$v_t + vv_x + Hv = 0$$

which behaves asymptotically as

$$v(x,t) = \frac{\mu}{2} + C|x - \mu t| \log(|x - \mu t|) + O(|x - \mu t| \log(|x - \mu t|)^{\frac{1}{2}})$$

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for some explicit C.

Main difficulties:

- Much more careful bounds needed: we need to work with $x \sim 10^{-10^6}$.
- Unclear what is the next term in the asymptotic expansion (even formally)

Back to Whitham: Uniqueness

Is the convex travelling wave that we found before the only solution?

- As in the case of Stokes wave, existence in the class of convex solutions does not imply uniqueness.

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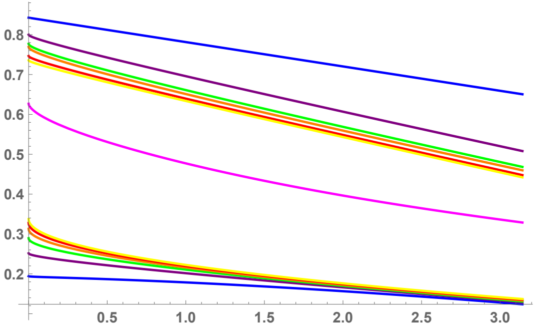
- There are $2\pi/k$-periodic solutions, so any attempt to prove uniqueness must rule out k-folds!
- Thus, we will consider the problem in the class of even, monotone solutions u which are increasing in $[0, \pi]$.
- *Monotonicity* + greatest height imply that solutions u are positive.

Uniqueness

Theorem (Enciso–JGS–Vergara, 2021)

The Whitham equation admits a unique, even, 2π-periodic traveling wave solution of greatest height between crest and trough that is non-increasing on $[0, \pi]$.



Uniqueness

- Main idea: obtain non-trivial upper and lower bounds u_0^+, u_0^- which are iteratively refined until they converge to the unique solution u of the equation:

$$u_0^- \leq u_1^- \leq \cdots \leq u_N^- \leq u \leq u_N^+ \leq \cdots \leq u_1^+ \leq u_0^+ \,,$$

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- The proof relies on very fine bounds for the dispersive multiplier as well as computer assisted estimates.

- The operator is not monotone and does not satisfy a maximum principle. We get contractivity only when we are sufficiently close to the unique solution u.

Uniqueness: setup

Derive estimates in $L^\infty(\mathbb{T})$ for the function $w(x) := |x|^{-1/2}u(x)$,

$$w^2 = \mathcal{F}w - \mathcal{G}w\,,$$

for positive (rather involved) linear operators $\mathcal{F}$ and $\mathcal{G}$:

$$\mathcal{F}(w)(x) := \frac{1}{|x|}\int_{y^*(x)}^{\pi} \mathcal{K}_x(y)\sqrt{|y|}w(y)\,dy\,,$$

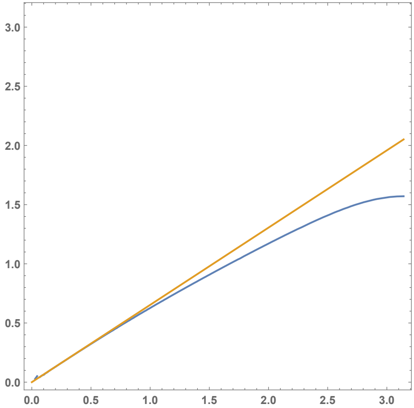
$$\mathcal{G}(w)(x) := \frac{1}{|x|}\int_{0}^{y^*(x)} |\mathcal{K}_x(y)|\sqrt{|y|}w(y)\,dy\,,$$

where

$$\mathcal{K}_x(y) = K(x-y) + K(x+y) - 2K(y)\,.$$

Uniqueness: setup

The kernel $\mathcal{K}_x(y)$ is positive when $y^*(x) < y < \pi$ and negative for $0 < y < y^*(x)$, with $y^*(x)$ a curve on $[0, \pi]$:



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1. First we prove rough initial bounds using fine estimates on the Whitham kernel.
2. Then we iterate those bounds using the monotonicity assumption until we can truly exploit the structure of the equation.
3. Finally we reach the regime in which we can make automatic iterations for a discrete (but large) approximation of our nonlinear system, plus small errors.

Uniqueness: initial bounds

- We exploit this behavior to obtain initial estimates in L^∞. For instance:

$$\|w\|_{L^\infty} \leq \|\mathcal{F}(1)\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \int_0^{1/r} \Big(\frac{1}{\sqrt{|1-t|}} + \frac{1}{\sqrt{1+t}} - 2 \Big) \cdot \frac{1}{t^2} \, dt + \text{ error} \,,$$

where r is the slope of the line tangent to the curve $y^*(x)$ at $x = 0$.

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- Asymptotic estimates yield $r = 0.652\ldots$ which in the end give us

$$w(x) \leq w_0^+(x) := 0.8425 + C|x| \,,$$

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- Analogously, we have the lower bound

$$w(x) \geq \frac{1}{\sqrt{\pi}} \left(2\sqrt{\delta} + \sqrt{2(1-\delta)} - \sqrt{2(1+\delta)} \right) \sqrt{|x|} - c|x| \,,$$

  Optimizing in δ:

$$w(x) \geq w_0^-(x) := 0.1940 - c|x| \,, \qquad c > 0 \,.$$

Uniqueness: self-improving bounds

- We have found then rough (but non-trivial!) bounds

$$w_0^-(x) \leq w(x) \leq w_0^+(x) \,.$$

- Define the operator $\mathcal{J} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$

$$\mathcal{J}(w^-, w^+)(x) := [\mathcal{F} w^-(x) - \mathcal{G} w^+(x)]^{1/2} \,.$$

  We would like to set up an iteration scheme that yields improved bounds

$$w_{n+1}^-(x) := \max\{w_n^-(x), \mathcal{J}(w_n^-, w_n^+)(x)\} \,,$$
$$w_{n+1}^+(x) := \min\{w_n^+(x), \mathcal{J}(w_n^+, w_n^-)(x)\} \,.$$

- However, $\mathcal{J}(w_0^-, w_0^+)(x)$ is not well defined for our initial bound. We need to work more!

Uniqueness: self-improving bounds

- There are threshold bounds $w^-_{n_0}(x)\,,\, w^+_{n_0}(x)$ for which the previous iteration scheme is well defined for all $n > n_0$.
- We introduce then a new operator $\widetilde{\mathcal{J}} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ that helps us to iterate lower bounds,

$$w^-_{n+1}(x) \geq \widetilde{\mathcal{J}}(w^-_n, w^+_n)(x)\,, \qquad 0 \leq n \leq n_0\,.$$

- This operator is crafted so that one can exploit the monotonicity in a clever way. In particular, it is build upon integral estimates for

$$K(\delta x - y) + K(\delta x + y) - K(x - y) - K(x + y)\,, \qquad 0 < \delta < 1\,.$$

- This procedure yields sharper bounds

$$w^-_{n_0+1}(x) = 0.3373 - 0.1172\sqrt{|x|} + 0.0023|x|\,,$$
$$w^+_{n_0+1}(x) = 0.7356 - 0.0194\sqrt{|x|} - 0.0824|x|\,.$$

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- Spatial discretization to approximate the operators $\mathcal{F}$ and $\mathcal{G}$ by $N \times N$ (interval) matrices

$$\mathcal{F}_{ij} := \mathcal{F}1_{(x_{j-1},x_j)}(x_i)\,, \qquad \mathcal{G}_{ij} := \mathcal{G}1_{(x_{j-1},x_j)}(x_i)\,,$$

and piecewise constant functions

$$w_n^-(x) := \sum_{j=1}^{N} w_{n;j}^- 1_{(x_{j-1},x_j)}(x)\,, \qquad w_n^+(x) = \sum_{j=1}^{N} w_{n;j-1}^+ 1_{(x_{j-1},x_j)}(x)\,,$$

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- Then:

$$\int_0^{x_2} \mathcal{K}_x(y)\sqrt{y}dy = \frac{1}{\pi}\left(f_{x_2,x} - f_{x_2,0}\right) + \text{small error}$$

where

$$f_{x_2,x_3} := \sum_{n=1}^{M_F} \frac{\cos(nx_3)}{n^2}\left[\sqrt{2\pi}\left(F_S(0) - F_S(\sqrt{\frac{2}{\pi}nx_2})\right)\right]$$
$$+ \sqrt{x_2}\left(S_{3/2}(x_2+x_3) + S_{3/2}(x_2-x_3)\right),$$

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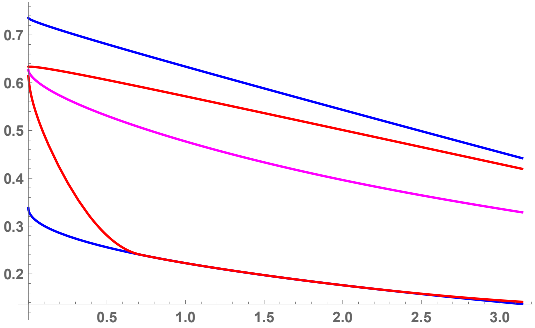
- Bonus: Other necessary integrals involving $\mathcal{K}_x$ use the ${}_2F_1$ hypergeometric for a faster calculation.

Uniqueness

We need one last pass to go from red to pink, changing our operators again and using the monotonicity.



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  so that $\sqrt{\mathcal{L}}$ is contractive provided that $\mathcal{C} > 0$ is a constant less than 1.

- Life is hard: $\mathcal{C} > 1$ if we work directly in X.

Uniqueness: fixed point argument

- Fortunately, there is room to circumvent this problem: $\sqrt{\mathcal{L}}$ becomes contractive in X endowed with the norm

$$\|u\|_X := \sup_{0<x<\pi} |x|^{-1/2} a^{-1}(x) |u(x)| \,, \qquad a(x) = 1 + 2\sqrt{|x|} \,.$$

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- This yields uniqueness in the class of even and monotone functions!

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- Moreover, this solution can be written as

$$v(x,t) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}}|x - \mu t|^{1/2} + \text{l. o. t.}$$

  where $\mu = 0.768\ldots$

THANK YOU!