ICERM mini course: Probabilistic well-posedness for nonlinear Schrödinger equation (I)

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1 Basic settings

1.1 A brief introduction of (NLS)

- $$\begin{cases} (i\partial_t+\Delta)u = \pm |u|^{p-1}u, & p\geq 3 \text{ odd} \\ u(0)=u_{in} \end{cases} \qquad \text{(NLS)}$$
  - We will focus on (NLS) on the torus $\mathbb{T}^d$ and its local (in time) theory.
  - Energy & mass conservation laws
  - "+" defocusing ; "−" focusing.

- Linear Schrödinger equation:

$$(i\partial_t+\Delta)u=0 \Rightarrow (i\partial_t - |k|^2)\hat{u}(t,k)=0 \Rightarrow \hat{u}(t,k)=e^{-it|k|^2}\hat{u}(0,k)$$

So, the linear solution can be written as $e^{it\Delta}u(0)$

  - $e^{it\Delta}$ preserves L^2 and H^s norms;

- Let's consider (NLS) in the $H^s(\mathbb{T}^d)$ spaces.

Q: For what s do we have Local wellposedness (LWP)?

Scaling argument for the threhold of s:

Suppose we have initial data

$$u(0) = f = \sum_{|k| \sim N} N^{-\alpha} e^{ik\cdot x}, \quad \alpha = s + \frac{d}{2}$$

then $\|f\|_{H^s} \sim 1$.

By Duhamel's formula:

$$u(t) = \underbrace{e^{it\Delta}}_{} f - i \int_0^t e^{i(t-s)\Delta} \left(|u(s)|^{p-1} u(s) \right) ds$$

- $\|e^{it\Delta} f\|_{H^s} = \|f\|_{H^s} \sim 1.$

- the second iteration

$$u^1(t) = -i \int_0^t e^{i(t-s)\Delta} \left(|e^{is\Delta} f|^{p-1} e^{is\Delta} f \right) ds$$

$$\hat{u}^1(t,k) = e^{-it|k|^2} \cdot \sum_{k = k_1 - k_2 + \cdots + k_p} \underbrace{\int_0^t e^{is\Omega} ds}_{\leq \frac{1}{\langle \Omega \rangle}} \cdot N^{-p\alpha}$$

where $\Omega = |k|^2 - |k_1|^2 + \cdots - |k_p|^2$ is the "resonance factor".

$$\Longrightarrow \hat{u}^1(t,k) \sim N^{-p\alpha} \cdot \sum_{\substack{k_1 - k_2 + \cdots + k_p = k \\ |k_j| \sim N}} \frac{1}{\langle \Omega \rangle}$$

$$\sim N^{-p\alpha} \cdot \sum_{k_1, \cdots, k_p,} h^b_{kk_1 \cdots k_p} \sim N^{-p\alpha} \cdot N^{pd-d-2}.$$

where the base tensor h^b is defined

$$h^b_{kk_1\cdots k_p} := \mathbb{1}\left\{\begin{array}{l} k_1-k_2+k_3-\cdots+k_p=k \\ \Omega = |k|^2-|k_1|^2+\cdots+|k_p|^2 = \text{const.}\end{array}\right\}.$$

we want $\|u^1(t)\|_{H^s} \lesssim 1.$

$$\Longleftrightarrow \left[\sum_{|k|\sim N}\left(N^s \cdot N^{-pd+pd-d-2}\right)^2\right]^{\frac{1}{2}} \lesssim 1$$

$$\Longleftrightarrow -pd+pd-d-2+s+\frac{d}{2} \leq 0$$

$$\Longleftrightarrow s \geq \underbrace{\frac{d}{2}-\frac{2}{p-1}}_{\substack{\shortparallel \\ s_{cr}}}$$

s_{cr} : (deterministic) scaling critical exponent for (NLS).

- This "s_{cr}" matches the threshold derived by

$$\left\| \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)\right\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|u\|_{\dot{H}^{s_c}(\mathbb{R}^d)}.$$

<u>THM 1.1</u>: Assume $s_{cr} \geq 0$, the (NLS) is LWP in H^s, if $s > s_c$, and is ill-posed if $s < s_{cr}$.

[Bourgain '93, Bourgain-Demeter '15]

(when $s = s_{cr} = 1$, LWP [Herr-Tataru-Tzvetkov '10])

Q: How does the "generic" data evolve in the spaces H^s $(s < s_{cr})$?

$\boxed{1.2}$ Random data theory $\left([\text{Bourgain } 96]^{\to \text{NLS}} \quad [\text{Burq - Tzvetkov } 08]_{\text{NLW}} \right)$

- Random initial data

$$u(0) = \varphi^{\omega} = \sum_{k \in \mathbb{Z}^d} \frac{g_k(\omega)}{\langle k \rangle^{\alpha}} e^{ik \cdot x} \qquad , \qquad \alpha = s + \frac{d}{2}$$

► $\{g_k\}_{k \in \mathbb{Z}^d}$ are i.i.d. Gaussians. $\mathbb{E}\, g_k = 0$

$\mathbb{E}\, |g_k|^2 = 1.$

► For all $k \in \mathbb{Z}^d$, A-certainly :

$$|g_k(\omega)| \lesssim A \log(\langle n \rangle + 1)$$

If some event happens with probability $\gtrsim 1 - C_\theta e^{-A^\theta}$, we call it "A-certainly".
(where θ is arbitrary small and A is a large parameter)

► Hence $\overset{a.s.}{\Rightarrow}$ $\varphi^{\omega} \in H^{s-}(\mathbb{T}^d) = \bigcap_{\varepsilon > 0} H^{s-\varepsilon}(\mathbb{T}^d).$

► When $\alpha = 1$, φ^{ω} is related to the invariant Gibbs measure. Formally

$$d\mu \sim \underbrace{\exp\left[-\frac{1}{p+1}\int_{\mathbb{T}^d} :|u|^{p+1}: dx\right]}_{\text{"weighte"}} \cdot \underbrace{\exp\left[-\frac{1}{2}\int_{\mathbb{T}^d} |\nabla u|^2 dx\right] \prod_{x\in\mathbb{T}^d} dx}_{\text{Gaussian measures}}$$

$$\left\{\frac{g_k}{\langle k\rangle}\right\}_{k\in\mathbb{Z}^d}$$

- Probabilistic scaling argument

  - Consider random data

$$u(0) = \varphi^\omega = N^{-\alpha}\sum_{|k|\sim N} g_k e^{ik\cdot x}, \qquad \alpha = s + \frac{d}{2}.$$

then $\|\varphi^\omega\|_{H^{s-}} \sim 1$. Second iteration u^1:

$$\hat{u}^1(t,k) = e^{-it|k|^2} \cdot N^{-p\alpha} \sum_{\substack{k_1-k_2+\cdots+k_p=k \\ |k_j|\sim N \\ \Omega = \text{Const.}}} g_{k_1}(\omega)\,\overline{g_{k_2}(\omega)}\cdots g_{k_p}(\omega)$$

  - If we assume $k_1\neq k_2$, $k_2\neq k_3$, etc. then these terms "orthogonal". This leads to square root cancellation

$$|\hat{u}^1(t,k)| \sim N^{-p\alpha}\cdot \Big(\underbrace{N^{pd-d-2}}_{\text{lattice counting}}\Big)^{\boxed{\frac{1}{2}}\,\leftarrow\,\text{large deviation}}$$

Then

$$\|\widehat{u^1}(t)\|_{H^{s}} \lesssim 1 \iff -p\alpha + \frac{1}{2}(pd-d-2) + s + \frac{d}{2} \leq 0$$

$$\iff \alpha \geq \frac{d}{2} - \frac{1}{p-1}$$

$$\iff s \geq \boxed{-\frac{1}{p-1}}$$

$\parallel$

s_{pr} : probabilistic scaling exponent for (NLS).

<u>Thm 1.2</u>: Assume $s > s_{pr}$, (NLS) is probabilistic LWP.
(Deng-Nahmod-Y., 2020). (a.s. LWP)

1.3 Large deviation property

<u>Lemma 1.3</u>:
$$\mathbb{P}\left(\omega : \left|\sum_{k\in\mathbb{Z}^d} a_k\, g_k(\omega)\right| > \lambda\right) \lesssim e^{-\frac{C\lambda^2}{\|a_k\|^2_{\ell^2_k}}}$$

where $\{a_k\}$ are constants in ℓ^2_k.

(Equivalently $\left|\sum_{k\in\mathbb{Z}^d} a_k\, g_k(\omega)\right| \lesssim A^{\theta} \cdot \|a_k\|_{\ell^2_k}$, A-certainly)

<u>Proof</u>: For $t > 0$ to be determined. (only $d=1$ case).

$$\int_{\Omega} e^{t\sum_{k\geq 1} a_k g_k(\omega)}\, d\mathbb{P}(\omega) \overset{\text{independence}}{=} \prod_{k\geq 1} \int_{\Omega} e^{t a_k g_k}\, d\mathbb{P}(\omega)$$

$$= \prod_{k\geq 1} \int_{-\infty}^{\infty} e^{t a_k \cdot x} \, d\mu_n(x) \leq \prod_{k\geq 1} e^{c(t a_k)^2} = e^{(ct^2)\sum_{k\geq 1} a_k^2}$$

$\uparrow$ Gaussian

Therefore, $$e^{(ct^2)\sum_{k\geq 1}(a_k)^2} \geq e^{t\lambda} P\Big(w : \sum_{k\geq 1} a_k g_k(w) > \lambda\Big)$$

$$\Longleftrightarrow \quad P\Big(w : \sum_{k\geq 1} a_k g_k(w) > \lambda\Big) \leq e^{(ct^2)\sum_k a_k^2} \cdot e^{-t\lambda}$$

By choosing $t = \dfrac{\lambda}{2c\sum_k a_k^2} \Rightarrow$

$$P\Big(w : \sum_{k\geq 1} a_k g_k(w) > \lambda\Big) \leq e^{-\frac{\lambda^2}{4c\|a_k\|_{\ell^2}^2}}$$

similarly we have also

$$P\Big(w : \sum_{k\geq 1} a_k g_k(w) < -\lambda\Big) \leq e^{-\frac{\lambda^2}{4c\|a_k\|_{\ell^2}^2}}.$$

$\square$

Lemma 1.4: $$F(w) = \sum_{\substack{k_1, k_2, k_3 \\ \cdots k_p}} a_{k_1, \cdots, k_p} \cdot g_{k_1} \overline{g_{k_2}} \cdots g_{k_p}$$

where $a_{k_1, \cdots, k_p}$ are coefficients in $\ell^2_{k_1, \cdots, k_p}$ and

$\{a_{k_1 \cdots k_p}\}$ is supported on $\left\{ \begin{array}{l} k_1 \neq k_2, k_4, \cdots \text{ and so on} \\ k_3 \neq k_2, k_4 \cdots \\ \cdots \end{array} \right\}$.

Similarly we also have A-certainly

$$|F(\omega)| \lesssim A^{\theta} \, \|a_{k_1 k_2 \cdots k_p}\|_{\ell^2_{k_1 \cdots k_p}}.$$

Proof: Skip. (Wiener chaos). □

1.4 Lattice Counting.

Lemma 1.5: (1) Let $R = \mathbb{Z}$ or $\mathbb{Z}[i]$. Then, given $0 \neq m \in R$ and $a_0, b_0 \in \mathbb{C}$, the number of choices for $(a, b) \in R^2$ that satisfy $m = ab$, $|a - a_0| \leq M$, $|b - b_0| \leq N$, is $O(M^{\theta} N^{\theta})$ with constant depending only on $\theta > 0$.

(2) Consider

const.

$$S = \left\{ (x, y, z) \in (\mathbb{Z}^2)^3 : \begin{array}{l} x - y + z = d \\ |x|^2 - |y|^2 + |z|^2 = \alpha \end{array} \quad \begin{array}{l} |x - a| \lesssim N_1 \\ |y - b| \lesssim N_2 \\ |z - c| \lesssim N_3 \end{array} \right\}$$

Then $\#S \lesssim \min\left((N_1 N_2)^{1+\theta}, (N_2 N_3)^{1+\theta}, (N_3 N_1)^{1+\theta} \right)$. $\leftarrow$ (in d-dimension case $(N_1 N_2)^{d-1+\theta}$ ...)

also $\#S \lesssim N_2^2 \cdot (\max(N_1, N_3))^{\theta}$.

Proof: (1) Standard divisor bounds: is $O(|m|^{\theta})$.

Now WLOG, suppose

otherwise $|m| \lesssim M^8$.

$$\max(|a_0|, M) \geq \max(|b_0|, N) \quad \text{and} \quad M_1 \sim |a_0| \gg M^4$$

hence $|m| \lesssim M_1^2$.

We then claim that the number of divisors a of m that satisfies $|a-a_0| \leq M$ is at most two.

In fact, suppose a, b, c are all in the ball $|x-a_0| \leq M$, then we have $\operatorname{lcm}(a,b,c) \mid m$, hence

$$\frac{abc}{\gcd(a,b)\gcd(b,c)\gcd(c,a)} \text{ divides } m.$$

Then $$M_1^2 \gtrsim |m| \geqslant \left| \frac{abc}{\gcd(a,b)\gcd(b,c)\gcd(c,a)} \right| \gtrsim M_1^3 M^{-3}$$

contradicting $M_1 \gg M^4$!

(2)

① $$2\langle d-x, y-x \rangle = |d|^2 - (|x|^2 - |y|^2 + |z|^2)$$

$$= |d|^2 - \alpha$$

$$(d_1-x_1)(y_1-x_1) + (d_2-x_2)(y_2-x_2) = \frac{|d|^2-\alpha}{2}$$

If $(d_1-x_1)(y_1-x_1) \neq 0$, we fix x_2, y_2 with $O(N_1 N_2)$ choices.

Then $(d_1-x_1)(y_1-x_1) = \text{const} \neq 0$.

$O(N_1^\theta N_2^\theta)$.

② Fix y with $O(N_2^2)$ then $x+z$ is fixed.

$x-z=w$

$$(w_1+iw_2)(w_1-iw_2) = |w|^2 = 2(|x|^2+|z|^2) - |x+z|^2$$

$$\Rightarrow (w_1, w_2) \text{ has } O(\max(N_1, N_3)^\theta).$$

$\square$

$\boxed{2}$ Bourgain's method [Bourgain '96]

$$\begin{cases} iu_t + \Delta u = :|u|^2 u: \\ u(0) = \varphi^\omega = \sum_{k\in\mathbb{Z}^2} \frac{g_k(\omega)}{\langle k\rangle} e^{ik\cdot x} \overset{a.s.}{\in} H^{0-} \end{cases}$$

↘ related to Gibbs measure

$\boxed{2.1}$ Wick ordering (renormalization)

$$:|u|^2 u: = |u|^2 u - 2\underbrace{\mathbb{E}\left(\int_{\mathbb{T}^2} |u|^2 dx\right)}_{\sum_{k\in\mathbb{Z}^2} \frac{1}{n^2} \to \infty} u$$

- Finite dimensional Approximation

$$\begin{cases} (i\partial_t+\Delta)\, u_N = \Pi_N\left(:|u_N|^2 u_N: \right) \\ u_N(0) = \Pi_N \varphi^\omega \end{cases} \qquad \text{(FDA)}$$

where $\widehat{\Pi_N f}(k) = \mathbb{1}_{\leq N}(k)\hat{f}(k)$. (and $\widehat{\Delta_N f}(k) = \mathbb{1}_{\frac{N}{2}<k\leq N}\hat{f}(k)$)

- Π_N can be replaced by any other Schartz function $\phi(\frac{k}{N})$ cutoffs.
- $:|u_N|^2 u_N:$ is also fine.
- $$:|u_N|^2 u_N: = \boxed{|u|^2 u - 2\left(\int_{\mathbb{T}^2} |u|^2 dx\right)\cdot u} + 2\underbrace{\left(\int_{\mathbb{T}^2} |u|^2 dx - \mathbb{E}\int_{\mathbb{T}^2} |u|^2 dx\right)}_{c_N}\cdot u$$

$$= \sum_{k\in\mathbb{Z}^2} \frac{|g_n|-1}{\langle k\rangle^2}\,, \quad \text{is uniformly bounded}$$

It is essentially that

$$\widehat{:|u_N|^2 u_N:}(k) \text{ can b } \sum_{\substack{k_1-k_2+k_3=k \\ |k_j|\leq N \\ k_2\neq k_1,k_3}} \hat{u}(k_1)\overline{\hat{u}(k_2)}\hat{u}(k_3)$$

no pairing

- Generally

$$\widehat{:|u_N|^{p-1} u_N:}(k) \text{ is } \sum_{\substack{k_1-k_2+k_3-\cdots+k_p=k \\ \text{no pairings}}} \hat{u}(k_1)\overline{\hat{u}(k_2)}\cdots\hat{u}(k_p).$$

$\boxed{2.2}$ Main theorem: (a.s. LWP) $\Longrightarrow$ invariant Gibbs measure & a.s. GWP.

$\{u_N\}$ τ^{-1}-certainly converges in $C_t^0 H_x^{0-}([0,\tau])$.

Or $\{u_N\}$ a.s. converges in $C_t^0 H_x^{0-}([0,T(\omega)])$ where $T(\omega)$ is a r.v. almost sure >0.

$\boxed{2.3}$ Bourgain's re-centering idea.

- The ansatz

$$u_N = \underbrace{e^{it\Delta}\Pi_N \varphi^\omega}_{\text{"R" for "random"}} + \underbrace{z_N}_{\text{"D" for "deterministic"}}$$

expected in H^s for some $s>0$.

- The equation for z.

$$\begin{cases} (i\partial_t + \Delta) D = \Pi_N\left(:|R+D|^2 (R+D): \right) =: \mathcal{N}(R+D) \\ z(0) = 0 \end{cases}$$

- The Duhamel form:

$$D(t) = i\int_0^t e^{i(t-s)\Delta}\left(\mathcal{N}(R+D)(s)\right) ds$$

Again,

$$\underbrace{\widehat{D}(t,k) \sim}_{\text{for simplicity}} \sum_{\substack{k_1-k_2+k_3=k \\ k_2\neq k_1,k_3 \\ |k_j|\leq N \\ |k|\leq N,\ \Omega=\text{Const.}}} \left(\widehat{R+D}(k_1)\ \overline{\widehat{R+D}(k_2)}\ \widehat{R+D}(k_3) \right)$$

$$:= \widehat{\mathcal{M}}_{np}(R+D \quad , R+D \quad , R+D)(k)$$

- After expanding the cubic and Littlewood-Paley decomposition,

$(R_{N_j} = \Delta_{N_j} R \;,\; D_{N_j} = \Delta_{N_j} R)$.

$R_{N_1}, R_{N_2}, R_{N_3} \leftarrow$ $N_1 \geq N_2 \geq N_3$, $N_2 \geq N_1 \geq N_3$, $N_3 \geq N_1 \geq N_2$, ⋮

$R_{N_1}, D_{N_2}, D_{N_3}$ ...

⋮

$D_{N_1}, D_{N_2}, D_{N_3}$ ...

Deterministic Local theory in H^s, $s > 0 = s_{cr}$.

Goal: $\|\mathcal{M}_{np}(\cdot, \cdot, \cdot)\|_{H^s} < \infty$.

2.4 Estimates.

Example: $(R_{N_1}, R_{N_2}, R_{N_3})$, $\quad N_1 \geq N_2 \geq N_3 \quad \|\mathcal{M}_{np}(\cdots)\|_{H^s}^2$

$$\sum_k \langle k\rangle^{2s} \left| \sum_{|k_j|\sim N_j} h^b_{kk_1k_2k_3} \cdot \frac{g_{k_1}}{\langle k_1\rangle} \cdot \frac{\overline{g_{k_2}}}{\langle k_2\rangle} \frac{g_{k_3}}{\langle k_3\rangle} \right|^2$$

Large deviation

$$\lesssim \sum_k \langle k\rangle^{2s} (N_1N_2N_3)^{-2} \cdot \left(\sum_{|k_j|\sim N_j} h^b_{kk_1k_2k_3} \right)$$

$$\leq (N_1)^{2s} (N_1N_2N_3)^{-2} \left(\sum_{\substack{k, \\ |k_j|\sim N_j}} h^b_{kk_1k_2k_3} \right)$$

$$\| $$

$$\# \left\{ \begin{matrix} k, k_1, k_2, k_3 : & k = k_1 - k_2 + k_3 \\ |k_j| \sim N & |k|^2 = |k_1|^2 - |k_2|^2 + |k_3|^2 \\ k_2 \neq k_1, k_3 & \end{matrix} \right\} \lesssim N_3^2 (N_1 N_2)^{1+\theta}$$

$$\lesssim \underset{\substack{\uparrow \\ \text{largest freq}}}{N_1}^{2(s-\frac{1}{2}+\theta)} \cdot N_2^{-1+\theta}$$

(required $s<\frac{1}{2}$).

$\boxed{2.5}$ Further discussions on $p>3$, odd cases.

$$\begin{cases} (i\partial_t+\Delta)u = |u|^{p-1}u, & x\in\mathbb{T}^2 \\ u(0)=\varphi^\omega = \sum_{k\in\mathbb{Z}^2} \frac{g_k}{\langle k\rangle} e^{ik\cdot x} \end{cases}$$

← related to Gibbs measure.

Q: Why Bourgain's re-centering idea cannot solve "invariant Gibbs measure for 2D (NLS) with $p>3$" ?

Recall: Bourgain's ansatz

$$u_N = \underset{\substack{\uparrow \\ \text{random,} \\ \text{a.s. in } H^{0-}}}{e^{it\Delta}\Pi_N\varphi^\omega} + \underset{\substack{\uparrow \\ H^s}}{z_N}$$

H^s, we treat it as a deterministic term.

- we need to make $z_N\in H^s$, $s>s_{cr}=\frac{d}{2}-\frac{2}{p-1} \overset{\mathbb{T}^2}{=} 1-\frac{2}{p-1}$

- consider a term. $N_1 \gg N_2 \sim N_3 \sim \cdots \sim N_p$

$$\left\| \mathcal{M}_{np}\left(R_{N_1}, R_{N_2}, R_{N_3}, \cdots, R_{N_p} \right) \right\|_{H^s}^2$$

$$= \sum_k \langle k \rangle^{2s} \left| \sum_{|k_j| \sim N_j} h^b_{kk_1k_2k_3} \cdot \frac{g_{k_1}}{\langle k_1 \rangle} \cdot \frac{\overline{g_{k_2}}}{\langle k_2 \rangle} \frac{g_{k_3}}{\langle k_3 \rangle} \cdots \frac{g_{k_p}}{\langle k_p \rangle} \right|^2$$

Large deviation

$$\lesssim \sum_k \langle k \rangle^{2s} (N_1 N_2 N_3 \cdots N_p)^{-2} \left(\sum_{|k_j| \sim N_j} h^b_{kk_1k_2k_3\cdots k_p} \right)$$

$$\lesssim (N_1)^{2s} (N_1 N_2 N_3 N_4 \cdots N_p)^{-2} \left(\sum_{\substack{k, \\ |k_j| \sim N_j}} h^b_{kk_1k_2k_3\cdots k_p} \right)$$

$$\# \left\{ \begin{array}{ll} k, k_1, k_2, k_3 : & k = k_1 - k_2 + k_3 - k_4 + \cdots k_p \\ |k_j| \sim N_j & |k|^2 = |k_1|^2 - |k_2|^2 + |k_3|^2 \\ & + \cdots + |k_p|^2 \end{array} \right\} \lesssim (N_3 N_4 \cdots N_p)^2 \cdot (N_1 N_2)^{1+\theta}$$

$$\lesssim N_1^{2(s-\frac{1}{2})+\theta} \cdot N_2^{-1+\theta}$$

Require $s < \frac{1}{2}$.

- However

$$s < \frac{1}{2} \leq s_c = 1 - \frac{2}{p-1}, \quad p \geq 5. \quad \boxed{!}$$

- In fact, all terms as

$$\mathcal{N}_{np}(R_{N_1}, u_L, \cdots, u_L) \quad \text{with } N_1 \gg L$$

are problematic!

Q: How to solve this problem?

- Put these high-low-low terms also in the center of ansatz.

$$u_N = \underbrace{e^{it\Delta}\Pi_N \varphi^{\omega}}_{H^{0-}} + \underbrace{(\text{high-low-low})_N}_{H^{\frac{1}{2}-}} + \underbrace{z_N}_{H^{1-}}$$

$H^{\frac{1}{2}-}$, with Random Averaging Operator Structure

3 Random Averaging Operator method

3.1 Main Thm

$$\begin{cases} (i\partial_t+\Delta)\, u_N = \Pi_N\left(:|u_N|^{p-1}u_N:\right) \\ u_N(0) = \Pi_N \varphi^\omega \end{cases} \qquad (\text{pNLS})$$

Theorem 3.1 (1) a.s. LWP

$\{u_N\}$ τ^{-1}-certainly converges in $C_t^0 H_x^{0-}([0,\tau])$ for (pNLS) on $\mathbb{T}^2$ $(p\geq 3,\ \text{odd})$

(2) Invariant Gibbs measure under the flow & a.s. GWP.

3.2 Ansatz

- Decompose the $\{u_N\}$, $y_N := u_N - u_{\frac{N}{2}}$.

$$(i\partial_t+\Delta)\, y_N = \Pi_N(\mathcal{N}(u_N)) - \Pi_{\frac{N}{2}}\left(\mathcal{N}(u_{\frac{N}{2}})\right)$$

$$= \Pi_N\left(\mathcal{N}(y_N + u_{\frac{N}{2}}) - \mathcal{N}(u_{\frac{N}{2}})\right)$$

$$+ \underbrace{\Delta_N\left(\mathcal{N}(u_{\frac{N}{2}})\right)}_{\text{Commutator term}}.$$

Recall:

$\Pi_N - \Pi_{\frac{N}{2}} = \Delta_N$

- Capture High-low-low terms. ("$L \ll N$")

$$\begin{cases} (i\partial_t+\Delta)\psi_{N,L} = \Pi_N \mathcal{N}(\psi_{N,L}, u_L, \cdots, u_L) \\ \psi_{N,L}(0) = \Delta_N \varphi^{\omega} \end{cases}$$

treat them as known. then it is just a linear equation.

k-th Fourier mode

$$(\psi_{N,L})_k = \sum_{\frac{N}{2}<|k^*|\le N} H^{N,L}_{kk^*} \frac{g_{k^*}}{\langle k^*\rangle}$$

For simplicity,

$$\psi_{N,\frac{1}{2}} = e^{it\Delta}\Delta_N \varphi^{\omega}$$

$$H^{N,\frac{1}{2}}_{kk^*} = e^{-i|k|^2 t}\mathbb{1}_{k=k_*}$$

- $H^{N,L}_{kk^*}$ is a random matrix totally depending on u_L which is a r.v. generated by evolving $\Pi_L \varphi^{\omega}$ ($\{g_k\}, |k|\le L$) under (pNLS).

We say $H^{N,L}_{kk^*} \in \mathcal{B}_{\le L}$

- Hense $H^{N,L}_{kk^*}$ is independent with $\frac{g_{k^*}}{\langle k^*\rangle}$ ($|k^*|\sim N$).

- $\zeta_{N,L} := \psi_{N,L} - \psi_{N,\frac{L}{2}}$

  $h^{N,L} := H^{N,L} - H^{N,\frac{L}{2}} \quad \in \mathcal{B}_{\leq L}$

- Full ansatz

$$\underbrace{y_N = \psi_{N,N^{1-\delta}} + z_N}$$

$$= \psi_{N,\frac{1}{2}} + \sum_{\substack{L \leq N^{1-\delta} \\ \uparrow \\ \text{dyadic}}} \zeta_{N,L} + z_N$$

$$(y_N)_k = \underbrace{e^{it\Delta} \Delta_N \varphi^\omega}_{H^{0-}} + \underbrace{\sum_{L \leq N^{1-\delta}} \left(\sum_{|k^*| \sim N} h^{N,L}_{kk^*} \frac{g_{k^*}}{\langle k^* \rangle} \right)}_{H^{\frac{1}{2}-}} + \underbrace{z_N}_{H^{1-}}$$

Then the equation for z_N :

Plug the ansatz into

$$(i\partial_t+\Delta)\,y_N = \Pi_N\Big(\mathcal{N}(y_N+u_{\frac{N}{2}})-\mathcal{N}(u_{\frac{N}{2}})\Big)$$

$$+\underbrace{\Delta_N\big(\mathcal{N}(u_{\frac{N}{2}})\big)}_{\text{Commutator term}}.$$

we have

$$(i\partial_t+\Delta)\,z_N = \Delta_N\Big(\mathcal{N}\big(\overbrace{\sum_{M\le\frac{N}{2}}\psi_{M,M^{1-\delta}}+z_M}^{u_{\frac{N}{2}}=\sum_{M\le\frac{N}{2}}y_N}\big)\Big)$$

$$+\Pi_N\Big(\mathcal{N}(z_N+\psi_{N,N^{1-\delta}}+u_{\frac{N}{2}})-\mathcal{N}(u_{\frac{N}{2}})$$

$$-\mathcal{N}(\psi_{N,N^{1-\delta}},u_{N^{1-\delta}},\cdots,u_{N^{1-\delta}})\Big)$$

- Bounds with the ansatz.

$$(y_N)_k = \underbrace{e^{it\Delta}\Delta_N\varphi^{\omega}}_{\uparrow\, H^{0-}} + \underbrace{\sum_{L\le N^{1-\delta}}\Big(\sum_{|k^*|\sim N} h^{N,L}_{kk^*}\frac{g_{k^*}}{\langle k^*\rangle}\Big)}_{\uparrow\, H^{\frac{1}{2}-}} + \underbrace{z_N}_{\uparrow\, H^{1-}}$$

where

- $\| h^{N,L}_{kk^*} \|_{k \to k^*} \lesssim L^{-\delta}$ (δ is a fixed small number)

- $\| h^{N,L}_{kk^*} \|_{\ell^2_{kk^*}} \lesssim N^{\frac{1}{2}+\delta} \cdot L^{-\frac{1}{2}}$

- $\left\| \left(1+ \frac{|k-k^*|}{L}\right)^{\kappa} h^{N,L}_{kk^*} \right\|_{\ell^2_{kk^*}} \lesssim N$ (κ is a very big number $\Downarrow$ "$|k-k^*| \lesssim L$")

- $\| (z_N)_k \|_{\ell^2_k} \lesssim N^{-1+\delta}$

Need to prove all these bounds with ansatz by induction on dyadic N.

3.3 Adapted large deviation & counting lemmas. & Estimates.

- Let's first select one estimate to be done.

$$\left\| \Delta_N \left(\mathcal{N}\Big(\underbrace{\sum_{M \le \frac{N}{2}} \psi_{M,M^{1-\delta}} + z_M}_{u_{\frac{N}{2}} = \sum_{M \le \frac{N}{2}} y_N} \Big) \right) \right\|_{\ell^2_k} \lesssim N^{-1+\delta}$$

one possible case is

$$\underbrace{\left\| \sum_{\substack{k_1,k_2,k_3 \\ k_2 \neq k_1,k_3 \\ |k_j| \sim N_j \leq \frac{N}{2} \\ \frac{N}{2} < |k| \leq N}} h^{b}_{kk_1k_2k_3} \cdot \left(\sum_{\substack{k_1^*,k_2^*,k_3^* \\ |k_j^*| \sim N_j}} h^{N_1,L_1}_{k_1k_1^*} \cdot \overline{h^{N_2,L_2}_{k_2k_2^*}} \, h^{N_3,L_3}_{k_3k_3^*} \times \frac{g_{k_1^*} \overline{g_{k_2^*}} g_{k_3^*}}{\langle k_1^* \rangle \langle k_2^* \rangle \langle k_3^* \rangle} \right) \right\|_{\ell^2_k}}$$

$$\left\| \sum_{\substack{k_1^*,k_2^*,k_3^* \\ |k_j^*| \sim N_j}} \tilde{h}_{kk_1^*k_2^*k_3^*} \cdot \frac{g_{k_1^*} \overline{g_{k_2^*}} g_{k_3^*}}{\langle k_1^* \rangle \langle k_2^* \rangle \langle k_3^* \rangle} \right\|_{\ell^2_k} \qquad (\dagger)$$

where

$$\tilde{h}_{kk_1^*k_2^*k_3^*} = \sum_{\substack{k_1,k_2,k_3 \\ k_2 \neq k_1,k_3 \\ |k_j| \sim N_j}} h^{b}_{kk_1k_2k_3} \cdot h^{N_1,L_1}_{k_1k_1^*} \cdot \overline{h^{N_2,L_2}_{k_2k_2^*}} \, h^{N_3,L_3}_{k_3k_3^*}$$

$$\left(\in \mathcal{B}_{\leq \frac{N}{2}} \right)$$

- First, consider the no-pairing case. $(k_2^* \neq k_1^*, k_3^*)$

By Lemma 3.2 (1)

$$(\dagger) \lesssim \Big\| \, \big\| \tilde{h}_{k k_1^* k_2^* k_3^*} \big\|_{k_1^* k_2^* k_3^*} \Big\|_{k} \cdot (N_1 N_2 N_3)^{-1}$$

$$\lesssim \Big\| \sum_{k_1, k_2, k_3} h^{b}_{k k_1 k_2 k_3} \cdot h^{N_1, L_1}_{k_1 k_1^*} \cdot \overline{h^{N_2, L_2}_{k_2 k_2^*}} \cdot h^{N_3, L_3}_{k_3 k_3^*} \Big\|_{k k_1^* k_2^* k_3^*} \cdot (N_1 N_2 N_3)^{-1}$$

$$\lesssim \big\| h^{N_1, L_1}_{k_1 k_1^*} \big\|_{k_1 \to k_1^*} \cdot \big\| h^{N_2, L_2}_{k_2 k_2^*} \big\|_{k_2 \to k_2^*} \cdot \big\| h^{N_3, L_3}_{k_3 k_3^*} \big\|_{k_3 \to k_3^*}$$

$$\times \big\| h^{b}_{k k_1 k_2 k_3} \big\|_{k k_1 k_2 k_3} \cdot (N_1 N_2 N_3)^{-1}$$

$$\lesssim L_1^{-\delta} L_2^{-\delta} L_3^{-\delta} \cdot N_1^{-1} N_2^{-1} N_3^{-1}$$

$$\# \underbrace{\left\{ \begin{array}{ll} (k, k_1, k_2, k_3) \in (\mathbb{Z}^2)^4 : & k - k_1 + k_2 - k_3 = 0 \\ & |k|^2 - |k_1|^2 + |k_2|^2 - |k_3|^2 = 0 \\ k_2 \neq k_1, k_3 & \\ k \neq k_1, k_3 & \frac{N}{2} < |k| \leq N \\ & \frac{N_j}{2} < |k_j| \leq N_j \end{array} \right\}^{\frac{1}{2}}}_{S :=}$$

WLOG assume $\frac{N}{2} \geqslant N_1 \geqslant N_2 \geqslant N_3$

(since $k - k_1 + k_2 - k_3 = 0 \Rightarrow N_1 \geqslant \frac{N}{6}$).

- Naive counting

$$(\dagger) \lesssim L_1^{-\delta} L_2^{-\delta} L_3^{-\delta} \cdot N_1^{-1} N_2^{-1} N_3^{-1} \cdot \left(N_3^2 \cdot (N_2 N_1)^{1+\theta} \right)^{\frac{1}{2}}$$

$$\lesssim L_1^{-\delta} L_2^{-\delta} L_3^{-\delta} \underbrace{N_1^{-\frac{1}{2}+\theta}}_{\substack{\uparrow \\ \text{but we need } N^{-1+\delta}}} \cdot N_2^{-\frac{1}{2}+\theta}.$$

- T-condition counting $\Rightarrow$ using Lemma 3.3
$|k|^2 \geqslant (\frac{N}{2})^2 \geqslant |k_1|^2$

$$(\dagger) \lesssim L_1^{-\delta} L_2^{-\delta} L_3^{-\delta} \cdot N_1^{-1} N_2^{-1} N_3^{-1} \cdot (N_2^2 \, N_3^2 \cdot N_1^{\theta})^{\frac{1}{2}}$$

$$\lesssim L_1^{-\delta} L_2^{-\delta} L_3^{-\delta} \, N_1^{-1+\theta} \lesssim N^{-1+\theta}$$

- Second, consider the pairing case. $(k_2^* = k_1^* \neq k_3^*)$

$(N_1 = N_2)$.

By Lemma 3.2 (2),

$$(\dagger) \lesssim$$

$$\Big\| \sum_{\substack{k_1, k_2, k_3 \\ k_1^*}} h^b_{k k_1 k_2 k_3} \cdot h^{N_1, L_1}_{k_1 k_1^*} \cdot \overline{h^{N_2, L_2}_{k_2 k_1^*}} \cdot h^{N_3, L_3}_{k_3 k_3^*} \Big\|_{k k_3^*} \cdot (N_1 N_2 N_3)^{-1}$$

$$\lesssim \underbrace{\Big\| h^b_{k k_1 k_2 k_3} \cdot \mathbb{1}_{|k_1 - k_2| \lesssim L_1 + L_2} \Big\|_{k k_1 k_2 k_3}}\cdot (N_1 N_2 N_3)^{-1}$$

$$\| h^{N_1, L_1}_{k_1 k_1^*} \|_{k_1 k_1^*} \; \| h^{N_2, L_2} \|_{k_1^* \to k_2} \cdot \| h^{N_3, L_3} \|_{k_3^* \to k_3}$$

Need to introduce new counting

$$\left\{ (k, k_1, k_2, k_3) \in (\mathbb{Z}^2)^4 : \begin{array}{l} k_1 - k_2 + k_3 = k \\ |k_1|^2 - |k_2|^2 + |k_3|^2 = |k|^2 \\ |k_j| \sim N_j \\ |k_1 - k_2| \lesssim L_1 + L_2 \end{array} \right\}$$

Lemma 3.2

(1) Assume $a_{k_1^* k_2^* k_3^*}(\omega)$ is independent with the Borel set generated by $\{g_k,\ k\in E\}$.

$$\sum_{\substack{k_1^*, k_2^*, k_3^* \\ k_j^* \in E \\ k_2^* \neq k_1^*, k_3^*}} a_{k_1^* k_2^* k_3^*}(\omega) \cdot g_{k_1^*} \overline{g_{k_2^*}} g_{k_3^*} = F(\omega)$$

no pairing

Then A-certainly we have

$$|F(\omega)| \lesssim A^{\theta} \cdot \left\| a_{k_1^* k_2^* k_3^*}(\omega) \right\|_{\ell^2_{k_1^* k_2^* k_3^*}}$$

(2) The same assumptions as (1)

$$\sum_{\substack{k_1^* = k_2^* \neq k_3^* \\ k_j^* \in E}} a_{k_1^* k_2^* k_3^*}(\omega) \cdot g_{k_1^*} \overline{g_{k_2^*}} g_{k_3^*} = F(\omega)$$

Then A-certainly we have

$$\left|F(\omega)\right| \lesssim A^{\theta} \left\| \sum_{k_1^* \in E} \mathbb{1}_{k_1^* = k_2^*} \cdot a_{k_1^* k_2^* k_3^*}(\omega) \right\|_{\ell^2_{k_3^*}} .$$

Lemma 3.3: (Γ-condition counting) ($N_1 \gtrsim N_2 \gtrsim N_3$)

$$\#\left(S \cap \left\{ |k|^2 \geq \underset{\substack{\uparrow \\ \text{fixed const.}}}{\Gamma} \geq |k_1|^2 \right\}\right) \lesssim N_2^2 \, N_3^2 \cdot N_1^\theta$$

Proof:

$$\begin{cases} k - k_1 + k_2 - k_3 = 0 \\ |k|^2 - |k_1|^2 + |k_2|^2 - |k_3|^2 = 0 \end{cases}$$

First, the choices of $|k_1|^2$ is bounded by

$$|k|^2 - |k_1|^2 = -|k_2|^2 + |k_3|^2 = (k_3 - k_2)(k_2 + k_3)$$

$$\lesssim \mathcal{O}(N_2^2)$$

Second, when $|k_1|^2$ is fixed, then the choices of k_1 is bounded by $\mathcal{O}(N_1^\theta)$.

Third, after k_1 is fixed, then the counting of (k, k_2, k_3), which is a three-vector counting, by Lemma 1.5 is bounded by $\mathcal{O}(N_3^2 \cdot N^\theta)$.

In total, it is $\mathcal{O}(N_2^2 N_3^2 N_1^\theta)$. $\square$