Ergodicity of Markov processes: theory and computation (2)

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September 9, 2021

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Last time

1. Markov processes on measurable state space.
2. Coupling method and renewal theory

Recall from last time

1. Φ_n – discrete time Markov process
2. (Φ_n^1, Φ_n^2) - coupled process. $\Phi_0^1 \sim \mu$, $\Phi_0^2 \sim \nu$.
3. α – atom. Coupling time τ_C – first simultaneous visit to α
4. Coupling lemma:

$$\|\mu P^n - \nu P^n\|_{TV} \leq 2\mathbb{P}[\tau_C > n]$$

5. Splitting method: create $\hat{\Phi}_n$ such that C_1 is an atom. If $\Phi_n \in C$, $\mathbb{P}[\hat{\Phi}_n \in C_1] = \delta$.

First simultaneous renewal time?

1. $S_n = Y_0 + Y_1 + \cdots + Y_n$, $S'_n = Y'_0 + Y'_1 + \cdots + Y'_n$
2. $Y_0 = \eta_\alpha|_{\Phi_0 \sim \mu}$, $Y'_0 = \eta_\alpha|_{\Phi_0 \sim \nu}$
3. $Y_1, Y'_1, Y_2, Y'_2, \cdots$ are i.i.d. with distribution $\eta_\alpha\|_{\Phi_0 = \alpha}$
4. Let T be the simultaneous renewal time

$$T = \inf_n \{ n = S_{k_1} = S'_{k_2} \text{ for some } k_1, k_2 \}$$

5. From renewal theorem: There exist n_0 and c such that

$$\mathbb{P}[n \text{ is a renewal time }] = \mathbb{P}[n = S_k \text{ for some } k] \geq c$$

for all $n \geq n_0$.

Theorems

Exponential tail

If $\mathbb{E}[\rho_1^{Y_0}], \mathbb{E}[\rho_1^{Y'_0}], \mathbb{E}[\rho_1^{Y_1}] < \infty$ for some $\rho_1 > 1$, then there exists $\rho_0 > 1$ such that $\mathbb{E}[\rho_0^T] < \infty$.

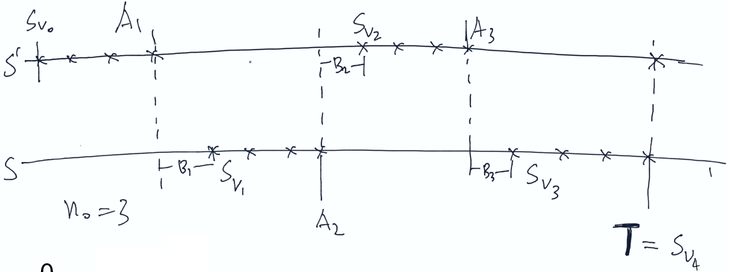
Power-law tail

If $\mathbb{E}[Y_0^\beta], \mathbb{E}[(Y'_0)^\beta], \mathbb{E}[Y_1^\beta] < \infty$ for some $\beta > 0$, then $\mathbb{E}[T^\beta] < \infty$.

(Note that finite exponential/power-law moment is equivalent to exponential/power-law tail.)

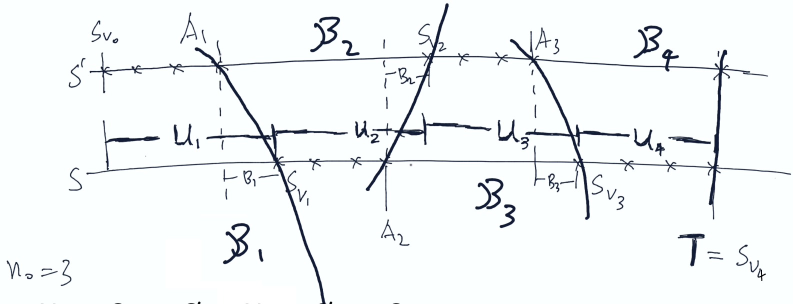
Ref: Lectures on the Coupling Method by Torgny Lindvall

Alternating σ-field (1)



1. $A_0 = 0$
2. $B_{2n} = \min\{S'_j - A_{2n} \mid S'_j - A_{2n} \geq 0\}$. First renewal after A_{2n} (denoted by $S'_{v_{2n}}$).
3. $A_{2n+1} = S'_{v_{2n}+n_0}$. Wait at least n_0 steps.
4. $B_{2n+1} = \min\{S_j - A_{2n+1} \mid S_j - A_{2n+1} \geq 0\}$. First renewal after A_{2n+1} (denoted by $S_{v_{2n+1}}$).
5. $A_{2n+2} = S_{v_{2n+1}+n_0}$

Alternating σ-field (2)



1. $U_1 = S_{v_1} - S'_{V_0}$, $U_2 = S'_{v_2} - S_{v_1}$, $\cdots\cdots$
2. Odd i
$$\mathcal{B}_i = \sigma\{Y_j, Y'_k \,|\, j \leq v_i, k \leq v_{i-1} + n_0\}$$
3. Even i
$$\mathcal{B}_i = \sigma\{Y_j, Y'_k \,|\, k \leq v_i, j \leq v_{i-1} + n_0\}$$

Random sum of random numbers

1. By renewal theorem, the probability of $B_k = 0$ is at least c.
2. Total number of attempts $\tau = \min\{k \mid B_k = 0\}$.
3. First simultaneous coupling time

$$T \leq B_0 + \sum_{i=1}^{\tau} U_i = Y_0 + \sum_{i=1}^{\infty} U_i \mathbf{1}_{\{\tau \geq i\}}$$

4. Now need to control moments of U_i.
5. See whiteboard for details.

How to move from τ_C to τ_α?

1. Let C be a small set with minorization condition

$$P(x, A) \geq \delta \mathbf{1}_C(x)\nu(A), \quad A \in \mathcal{B}(X)$$

2. Split C into C_0 and C_1
3. Every time when Φ_n visits C, $\hat{\Phi}_n$ has probability δ to visit C_1 (the atom).
4. If we have either $\mathbb{E}[\rho^{\eta_C}] < \infty$ for some $\rho > 1$ (exponential) or $\mathbb{E}[\eta_C^\beta] < \infty$ (power-law), what can we say about η_α?
5. Same idea: random sum of random numbers.

Random sum of random numbers (again!)

1. Small set C with $P(x, A) \geq \delta \mathbf{1}_C \nu(A)$
2. $\tau^1 = 0$, $\tau^1 = \eta_C$ first passage time to C
3. $\tau^{n+1} = \inf\{n \,|\, \Phi_n \in C, n \geq \tau^n\}$ $n+1$-th passage time to C
4. $Z_n = 1$ if $\Phi_{\tau^n} = \alpha$, $Z_n = 0$ otherwise.
5. Z_n is measurable on $\sigma(\Phi_0, \cdots, \Phi_{\tau^n})$
6. $P_x[Z_n = 1 \,|\, \mathcal{F}_{\tau^{n-1}}] = \delta > 0$
7. Let $\xi = \inf\{n \,|\, Z_n = 1\}$. Number of visiting to C needed to enter α. $\eta_\alpha = \tau^\xi$.

Theorems

Exponential tail case

If $\sup_{x \in C} \mathbb{E}_x[r^{\tau^1}] < \infty$ and $\mathbb{E}_\mu[r^{\tau^1}] < \infty$ for some $r > 1$, then there exists $r_1 > 1$ such that $\mathbb{E}_\mu[r_1^{\tau^\xi}] < \infty$ and $\sup_{x \in C} \mathbb{E}_x[r_1^{\tau^\xi}] < \infty$.

Power-law tail case

Let $\hat{\alpha}_k = 2, 2, 3, 4, 5, \cdots$.

Let $f_\beta(n) = \sum_{k=1}^{n} \hat{\alpha}_k{}^n$. Then there exists C_β such that $C_\beta^{-1} n^\beta \leq f_\beta(n) \leq C_\beta n^\beta$.

If $\sup_{x \in C} \mathbb{E}_x[f_\beta(\tau^1)] < \infty$ and $\mathbb{E}_\mu[f_\beta(\tau^1)] < \infty$ for some $\beta > 0$, then $\mathbb{E}_\mu[f_\beta(\tau^\xi)] < \infty$ and $\sup_{x \in C} \mathbb{E}_x[f_\beta(\tau^\xi)] < \infty$.

Proof on the white board

Criterion for ergodicity

1. Find a small set C.
2. Split the small set to get an atom α.
3. Independent coupling. Couple at time T (first simultaneous visit to the atom).
4. Random sum of random numbers episode 1: exponential/power-law tail of η_C gives exponential/power-law tail of η_α
5. Random sum of random numbers episode 2: exponential/power-law tail of η_α gives exponential/power-law tail of T

Question: First passage time to the small set?

Thank you