RELATIVISTIC FLUID EQUATIONS

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1. Introduction

This note is a compilation from [1, 2, 3, 4, 6]. We also point out [9] as nice references.

1.1. **Notations.**

We consider Minkowski space $(\mathbb{R}^{1+3}, g_{\alpha\beta})$ with $g_{00} = -c^2$, $g_{ij} = \delta_{ij}$ and $g_{0j} = g_{j0} = 0$. Its inverse is denoted $g^{\mu\nu}$ where $g^{00} = -c^{-2}$, $g^{ij} = \delta_{ij}$ and $g^{0j} = 0 = g^{j0}$. We use the Einstein convention that repeated up-down indices be summed and we raise and lower indices using the metric as follows:

$$A^\alpha = g^{\alpha\beta} A_\beta, \quad B_\alpha = g_{\alpha\beta} B^\beta.$$

In addition, latin indices $i, j \dots$ vary from 1 to 3, while greek indices $\mu, \nu \dots$ vary from 0 to 3.

Note in particular that the coordinate along u^ν is given by $c_\nu = |u^\mu u_\mu|^{-\frac{1}{2}} u_\nu$ and that the projection along u is given by[1]

$$(\mathrm{proj}_u)^\mu{}_\nu = \frac{1}{u^\alpha u_\alpha} u^\mu u_\nu.$$

We denote $\mathcal{T}^d(M)$ the set of contravariant d-tensors on the Minkowski space.

1.1.1. *Poincaré group.*

The Poincaré group is the group of isometries of (the affine space) M. Besides translations, we have the Lorentz transformations $\mathcal{O}(3,1)$, i.e. the set of linear transformations L such that

$$g(LX, LX) = g(X, X),$$

or in other words, $L_{\alpha\beta} L^{\alpha\gamma} = \delta_\beta^\gamma$.

The first postulate is that all the laws of special relativity should be invariant (in fact co-variant) under Poincaré transformations[2].

1.2. **Vocabulary.**

The vectors of M are naturally separated into

- *time-like* vectors v^ν such that $g_{\mu\nu} v^\mu v^\nu < 0$ (just as for ∂_t).
- *null* vectors for which $g_{\mu\nu} v^\mu v^\nu = 0$.
- *space-like* vectors for which $g_{\mu\nu} v^\mu v^\nu > 0$ (just as ∂_i).

In fact, one can also look at causality to define 6 different types of vectors[3] (future-oriented time-like, future-oriented null, zero, space-like, past-oriented null and past-oriented time-like). We can easily see that all of the above categories are invariant under Lorentz transform.

[1] The tensors with up indices are vectors; the tensor with down indices are forms. While the use of the metric allows to largely identify the two (especially in special relativity), it is often convenient to keep in mind which objects are naturally vectors and which are naturally 1-forms. For example, the 4-velocity is naturally a vector, while the momentum will naturally be a 1-form.

[2] Another way of stating this is that the laws of special relativity should be the same when expressed in any *inertial frame*, i.e. in any coordinate frame which is the image of the standard coordinate frame $((0,0,0,0), \partial_t, \partial_1, \partial_2, \partial_3)$ under a Poincaré transform

[3] The six different types are the cosets under dilation and isometries. As I understand this, the idea is that dilating amounts to choosing a scale; an isometry amounts to choosing an admissible frame; the relevant information should be independent of these two operations. For comparison, in the Euclidean space, there are only 2 different equivalence class: $\{0\}$ and nonzero vectors. In this sense, even at this rough level, the geometry of Minkowski space is much richer.

In the Minkowski space, there is no notion of absolute time and *the time axis depends on the observer.* We define *an event* to be a point (t, x) of M. We define an *observer* to be a point and a time axis v (defined by a time-like vector v). We define the *rest space* (or *simultaneity space*) of an observer to be the 3-space $(t, x) + v^{\perp}$. *Thus, now the notion of simultaneity depends on the observer.*

Note that if $v = p - q$ is (future-oriented) time-like or null, then, for all the observers, q precedes p, whereas if $p - q$ is space-like, then there are observers for which p precedes q and some for which q precedes p.

1.2.1. *Point particles.* A particle in motion is then only described by its curve $p(s) : \mathbb{R} \to M$, its *world-line*. Physically, we only consider particle such that the tangent vector $\dot{p} = \partial_s p$ is time-like. Since only the curve is relevant, we might as well parameterize it by arc length. Thus, from now on, we assume that for any world line p, we have that[4]

$$g_{\alpha\beta}\dot{p}^{\alpha}\dot{p}^{\beta} = -c^2.$$

1.2.2. *Stress-energy tensor (or energy-momentum tensor).* In relativity, the properties of a matter field are all summarized in a *stress-energy tensor* $T^{\mu\nu} \in \mathcal{T}^2(M)$, which is symmetric $T^{\mu\nu} = T^{\nu\mu}$ and positive in the sense that for any time-like vector v, we have that $T^{\mu\nu}v^{\mu}v^{\nu} \geq 0$. This tensor is defined from the *energy*, *momentum* and *stress* of the matter in the following way: for any observer with axis v[5],

- The energy of the matter field he measures is given by $\epsilon = c^{-2}T^{\mu\nu}v_{\mu}v_{\nu}$,
- The linear momentum density he measures is given by $p^{\nu} = -c^{-2}T^{\mu\nu}v_{\mu}$,
- The matter stress tensor that he measures is given by $S = T_{|v^{\perp}}$.

In *general relativity*, the *total stress-energy tensor* T (given by the *sum* of all the stress-energy tensors of each matter field present) determines the metric through the *Einstein equations*:

$$E_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.1}$$

where $E = \mathrm{Ric} - \frac{1}{2}Rg$ is the *Einstein tensor of the metric*, Λ is the cosmological constant and G is the gravity constant (coming from Newtonian theory). In particular, from the *Bianchi identity* we obtain that

$$\nabla_{\nu}T^{\mu\nu} = 0. \tag{1.2}$$

where ∇ denotes the covariant derivative.

In *special relativity*, we *assume* that $E = \Lambda = 0$ here and we neglect (1.1). However, it is desirable that we retain (1.2) for the total stress-energy tensor (in this case, the covariant derivative becomes a simple partial derivative).

1.3. **Perfect fluid.** A *perfect fluid* or *simple fluid* is a (relativistic) model for a fluid in which the fluid is modeled by

- a 4-velocity $u^{\nu} \in \mathcal{T}^1(M)$ whose integral curve give the world-line of all the particles of fluid. It satisfies
$$u_{\mu}u^{\mu} = -c^2; \tag{1.3}$$
- a density function $n \in \mathcal{T}^0(M)$ such that $n(t, x)$ gives the density of particles at the event (t, x)
- an energy-density ϵ which gives the (density of) total energy of a fluid particle *at rest*
- an *isotropic* stress in its rest frame given by pI_3.

[4]Sometimes the world-line is parameterized by $g_{\alpha\beta}\dot{p}^{\alpha}\dot{p}^{\beta} = -mc^2$ instead, but when we have several fluids, we find it better to use this notation.

[5]The formula above are better understood if we keep in mind that $c^{-1}v_{\mu}$ is normalized.

From all these assumptions, we see that the stress energy-momentum is then given by

$$T^{\mu\nu} = (\epsilon + p)\frac{u^\mu u^\nu}{c^2} + pg^{\mu\nu} = \epsilon\frac{u^\mu u^\nu}{c^2} + p\eta^{\mu\nu}, \tag{1.4}$$

where $\eta^{\mu\nu} = g^{\mu\nu} + c^{-2}u^\mu u^\nu$ is the (Euclidean) metric on the rest-space of u.

The precise form of ϵ, p depends on thermodynamics assumptions.

1.3.1. *Other noteworthy types of fluids.* A *dust* is a pressureless perfect fluid; thus its stress-energy tensor is given by

$$T^{\mu\nu} = \rho\frac{u^\mu u^\nu}{c^2}.$$

A *radiation field* is a perfect fluid for which $\epsilon = 3p$; thus its stress-energy tensor is given by

$$T^{\mu\nu} = p\left[4\frac{u^\mu u^\nu}{c^2} + g^{\mu\nu}\right].$$

1.4. **Thermodynamics.** The *first principle of thermodynamics* states that there exists an equation of states[6]:

$$\begin{aligned} &d\{\epsilon V\} = Td\{sV\} + \mu d\{nV\} - pdV \qquad (d\mathcal{E} = TdS + \mu dN - pdV),\\ &d\epsilon = Tds + \mu dn, \quad p = Ts + \mu n - \epsilon, \end{aligned} \tag{1.5}$$

where

- T denotes the temperature,
- s denotes the entropy,
- μ denotes the chemical potential of the fluid,
- V denotes any volume.

One can understand these equations by saying that there is an equation of state $\epsilon = \epsilon(s, n)$ depending on the density and the entropy, and this defines the temperature and chemical potential by

$$T = \frac{\partial \epsilon}{\partial s}, \quad \mu = \frac{\partial \epsilon}{\partial n}. \tag{1.6}$$

Note that ϵ denotes the total energy, $\epsilon = nmc^2 + \epsilon_{int}$ where ϵ_{int} accounts for the internal energy (including the microscopic energy density, the potential energy density from microscopic interactions. . .). Therefore, we see that $\mu = mc^2 + \mu_{int}$.

1.5. **Dynamics for one neutral fluid.** For simplicity, we start with the simpler case of a *simple* neutral fluid. In this case, we can derive the equations of motion for a fluid starting from 2 "first principles".

- The particle are conserved (neither created nor annihilated), in which case

$$\partial_\nu(nu^\nu) = 0. \tag{1.7}$$

- The Bianchi identity (1.2) holds

$$\forall\mu, \quad \partial_\nu\left[(\epsilon + p)\frac{u^\mu u^\nu}{c^2} + pg^{\mu\nu}\right] = 0.$$

Note that (1.3), (1.7) and (1.2) constitute 6 equations for the 6 unknowns u^ν, n, s.

It is convenient to introduce the *enthalpy* h defined by

$$nh = \epsilon + p = Ts + \mu n, \tag{1.8}$$

in which case, using (1.7), we see that the equation above simplifies to

$$\frac{nu^\nu}{c^2}\partial_\nu\left[hu^\mu\right] + g^{\mu\nu}\partial_\nu p = 0. \tag{1.9}$$

[6] Or that we can describe the energy solely in terms of the density and the entropy.

We claim that (1.7) and (1.9) gives the equation of motion.

1.5.1. *Dynamics along u.* Projecting (1.9) parallelly to u gives

$$0 = h\frac{nu^\nu}{c^2}u_\mu\partial_\nu u^\mu + \frac{nu^\nu}{c^2}u^\mu u_\mu\partial_\nu h + u_\mu g^{\mu\nu}\partial_\nu p = 0.$$

The first term vanishes in view of (1.3). We then get

$$u^\nu\left[\partial_\nu p - n\partial_\nu h\right] = 0 \tag{1.10}$$

Using (1.8) and (1.6), we see that

$$\partial_\nu p - n\partial_\nu h = (h-\mu)\partial_\nu n - T\partial_\nu s = \frac{sT}{n}\partial_\nu n - T\partial_\nu s$$

and therefore, using once more (1.7),

$$0 = \frac{sT}{n}u^\nu\partial_\nu n - Tu^\nu\partial_\nu s = -T\partial_\nu\left[su^\nu\right].$$

The *third law of thermodynamics* implies that

$$T \geq 0, \qquad T = 0 \Rightarrow s = 0,$$

in which case we find that

$$\partial_\nu\left[su^\nu\right] = 0.$$

Now, if we define

$$\overline{s} = \frac{s}{n},$$

we finally obtain that

$$u^\nu\partial_\nu\overline{s} = 0. \tag{1.11}$$

Therefore, if we assume that at initial time, each particle carries the same entropy, $\overline{s} \equiv \overline{s}_0$, then this remains true for all times, and we have $\overline{s} \equiv \overline{s}_0$ uniformly in space and time. Thus, coming back to the equation of state, se that in this case $\epsilon = \epsilon(n)$ and we say that the fluid is *barotropic*.

For a barotropic fluid, (1.8) gives that

$$\frac{dh}{dn} = \frac{1}{n}\frac{dp}{dn}$$

and we see that the equation (1.9) along u is trivially satisfied. To conclude: *if we consider a simple neutral fluid such that, at initial time $\overline{s} \equiv \overline{s}_0$, then this is propagated by the flow, h and p depend only on n, and the parallel component of* (1.9) *along u is satisfied.*

Definition: A simple fluid is called *isentropic* if at some initial time $\overline{s} \equiv \overline{s}_0$, in which case, this remains always true.

1.5.2. *Dynamical equations for barotropic fluids.* For barotropic fluids, the fluid is fully determined by only n and u^ν satisfying (1.3). These are 4 unknowns and they satisfy (1.7) and the part of (1.9) orthogonal to u. These are 4 equations. Projecting (1.9) in the direction orthogonal to u gives

$$\frac{nh}{c^2}a^\mu + \eta^{\mu\nu}\partial_\nu p = 0, \quad \eta^{\mu\nu} = \frac{u^\mu u^\nu}{c^2} + g^{\mu\nu},$$

where $a^\mu = u^\nu\partial_\nu u^\mu$ denotes the acceleration and η the restriction of the metric to the rest frame. Therefore the equations of motion are

$$\begin{aligned}\partial_\nu(nu^\nu) &= 0,\\ a^\mu &= -c^2\eta^{\mu\nu}\partial_\nu\left[\ln h\right].\end{aligned} \tag{1.12}$$

1.5.3. *Vorticity.* An important quantity is the relativistic momentum

$$\pi_\alpha = c^{-2} h u_\alpha, \tag{1.13}$$

which allows to define the relativistic vorticity

$$\omega_{\alpha\beta} = \partial_\alpha \pi_\beta - \partial_\beta \pi_\alpha \qquad \text{i.e. } \omega = d\pi.$$

We then obtain the *Lichnerowicz* equation of motion

$$\partial_\nu(n u^\nu) = 0, \quad u^\alpha \omega_{\alpha\beta} = T\partial_\beta \overline{s}.$$

Which simplifies in the isentropic case to the *Synge relation*

$$u^\alpha \omega_{\alpha\beta} = 0.$$

1.5.4. *Irrotational flows.* Consider an *isentropic* flow. In this case, we see from (1.9) that

$$\begin{aligned} u^\nu \partial_\nu \partial_\alpha [h u_\beta] &= \partial_\alpha [u^\nu \partial_\nu [h u_\beta]] - (\partial_\alpha u^\nu)\partial_\nu [h u_\beta] \\ &= \partial_\alpha \left[-g_\beta^\theta \frac{c^2}{n} \frac{dp}{dn} \partial_\theta n \right] - (\partial_\alpha u^\nu)\omega_{\nu\beta} - (\partial_\alpha u^\nu)\partial_\beta [h u_\nu] \\ &= -c^2 \partial_{\alpha\beta} h - (\partial_\alpha u^\nu)\omega_{\nu\beta} - (\partial_\alpha u^\nu) h (\partial_\beta u_\nu) \end{aligned}$$

and therefore, we see that the vorticity is transported in the sense that

$$u^\nu \partial_\nu \omega_{\alpha\beta} = (\partial_\alpha u^\nu)\omega_{\beta\nu} - (\partial_\beta u^\nu)\omega_{\alpha\nu}. \tag{1.14}$$

Consequently, if an isentropic flow satisfies $\omega \equiv 0$ at initial time, it remains so for all later time.

Reciprocally, it follows from the Lichnerowicz equation and the third law of thermodynamics that a flow which is irrotational is isentropic.

An isentropic and irrotational flow is a potential flow in the sense that there exists a function $\phi : M \to \mathbb{R}$ such that

$$\pi_\alpha = c^{-2} h u_\alpha = \partial_\alpha \phi.$$

The equations of motion then give, after some computations that

$$\Box_g \phi - g^{\mu\nu} \partial_\mu \phi \partial_\nu \left\{ \ln \frac{h}{n} \right\} = 0, \qquad \partial_\mu \left\{ \frac{\partial_\alpha \phi \partial^\alpha \phi}{2} + c^2 \ln h \right\} = 0,$$

which seems similar to a relativistic potential flow equation.

1.6. **Lorentz Covariance.** Consider a Lorentz-transformation L, i.e. a (fixed) 2-tensor L satisfying $L_{\alpha\beta} L^{\alpha\gamma} = \delta_\beta^\gamma$ and define[7]

$$(X')^\alpha = L^{\alpha\beta} X_\beta, \quad n'(X') = n(X), \quad h'(X') = h(X), \quad \epsilon'(X') = \epsilon'(X), \quad (u')^\alpha(X') = L^{\alpha\beta} u_\beta(X)$$

Then, we see that (n, h, ϵ, u) satisfy (1.7) and (1.9) if and only if (n', h', ϵ', u') does.

For the sake of example, let us do the computations in detail.

We first see that $X^\gamma = L_\alpha{}^\gamma (X')^\alpha$. Denoting ∂' the derivative in the new coordinates, we compute

$$\begin{aligned} \partial'_\nu [n'(X')\,(u')^\nu(X')] &= L^{\nu\mu} \partial'_\nu [n(X) u_\mu(X)] \\ &= L^{\nu\mu} \partial_\omega [n u_\mu](X) \cdot \partial'_\nu (X^\omega) \\ &= L^{\nu\mu} L_\theta{}^\omega \delta_\nu^\theta \partial_\omega [n u_\mu](X) = 0. \end{aligned}$$

[7]Of course, we keep g unchanged.

Similarly for the second equation,

$$\begin{aligned}\frac{n'(u')^\nu}{c^2}\partial'_\nu[h'(u')^\mu]+g^{\mu\nu}\partial'_\nu p' &= L^{\nu\alpha}L^{\mu\beta}\frac{nu_\alpha(X)}{c^2}\partial'_\nu[hu_\beta(X)]+g^{\mu\nu}\partial'_\nu p(L_\alpha{}^\gamma(X')^\alpha)\\ &= L^{\nu\alpha}L^{\mu\beta}\frac{nu_\alpha(X)}{c^2}\partial_\theta[hu_\beta](X)\cdot\partial'_\nu(L_\omega{}^\theta(X')^\omega)+g^{\mu\nu}\partial_\theta p\partial'_\nu(L_\omega{}^\theta(X')^\omega)\\ &= L^\nu{}_\alpha L^{\mu\beta}L_\omega{}^\theta\delta^\omega_\nu\frac{nu^\alpha}{c^2}\partial_\theta[hu_\beta]+L_\omega{}^\theta\delta^\omega_\nu g^{\mu\nu}\partial_\theta p=0.\end{aligned}$$

1.7. **The Crocco equation.**

1.8. **Lagrangian.** There is also a Lagrangian formulation, but I could not find one that I thought was completely satisfactory. We give here one Lagrangian from [8] seems to work but we will not dwell on this.:

$$S=\frac{1}{c}\int_M d^4x\sqrt{-\det g}\left\{-\frac{1}{2\kappa}R-\rho(\epsilon+c^2)-\frac{1}{2}\mu_1\left(u_\mu u^\nu+c^2\right)+\mu_2\frac{d\sigma}{d\lambda}\right\}$$

where $\kappa=8\pi G$, λ is the parameter for the curve x^μ, $u^\mu=\frac{dx^\mu}{d\lambda}$, σ is the entropy density and μ_1 and μ_2 are Lagrange multipliers.

1.9. **Newtonian approximation.** In this section, we see how the general relativity comes into play in the equation for the dynamics of fluids, at first order (i.e. under the *weak field approximation* that the metric is a small deviation from the Minkowski metric and the *small velocity approximation* where the perfect fluid is close to being stationary). This allows to verify the normalization of our constants. We refer to [7] for a more rigorous result. In the following, we consider a single barotropic fluid.

We assume that the cosmological constant vanishes, $\Lambda=0$. In this case, upon taking the trace of (1.1), we obtain that

$$R=-\frac{8\pi G}{c^4}\mathcal{T},\quad \mathcal{T}=g_{\mu\nu}T^{\mu\nu}=-(\epsilon-3p).$$

Thus we may rewrite (1.1) as

$$\mathrm{Ric}_{\mu\nu}=\frac{8\pi G}{c^4}\left\{T_{\mu\nu}-\frac{1}{2}\mathcal{T}g_{\mu\nu}\right\}=\frac{8\pi G}{c^4}\left\{\frac{\epsilon+p}{c^2}u_\mu u_\nu+\frac{\epsilon-p}{2}g_{\mu\nu}\right\} \tag{1.15}$$

Now, we assume that $g_{\mu\nu}$ is a small variation of the Minkowski metric which (only in this section) we write as $m_{\mu\nu}$. Thus $g_{\mu\nu}=m_{\mu\nu}+\theta_{\mu\nu}$ where (in Minkowski-geodesic coordinates), $|\theta|\ll 1$ and $\theta\to 0$ at spatial infinity (in the Minkowski variables). Then (1.15) and (3.1) give, in g-harmonic coordinates,

$$\frac{1}{2}\Box_g\theta_{\mu\nu}=\frac{8\pi G}{c^4}\left\{\frac{\epsilon+p}{c^2}u_\mu u_\nu+\frac{\epsilon-p}{2}m_{\mu\nu}+\frac{\epsilon-p}{2}\theta_{\mu\nu}\right\}+O(\partial\theta)^2.$$

Now, we expand

$$\frac{1}{2}\Box_g\theta_{00}=\frac{8\pi G}{c^4}\left\{\frac{\epsilon+p}{c^2}c^4\gamma^2-\frac{\epsilon-p}{2}c^2+\frac{\epsilon-p}{2}\theta_{00}\right\}+O(\partial\theta)^2$$

$$\frac{1}{2}\left\{\frac{1}{c^2}\partial_{tt}-\Delta\right\}\theta_{00}-\frac{\epsilon-p}{2c^2}\frac{8\pi G}{c^2}\theta_{00}=4\pi G\left\{\frac{\epsilon+3p}{2c^2}+\frac{\epsilon+p}{2c^2}\frac{|v|^2}{c^2}+\frac{\epsilon+p}{c^2}O(\frac{|v|^2}{c^2})^2\right\}+O((\partial\theta)^2+\theta\partial^2\theta_{00}).$$

Assuming that

$$u^\nu=\gamma(1,v^j),\quad |v|\ll c,\quad \|\theta\|_{C^{10}}\ll 1,\quad \frac{\epsilon}{c^2}=nm+O(\frac{1}{c^2}),\quad p=O(1),$$

we see that the equation above reduces to

$$-\Delta\theta_{00}=8\pi Gnm+O(\frac{1}{c^2}+\frac{|v|^2}{c^2}+\|\theta\|^2_{C^2}). \tag{1.16}$$

For the other coordinates, we find that

$$\frac{1}{2}\Box_g\theta_{0j} - \frac{\epsilon - p}{2c^2}\frac{8\pi G}{c^2}\theta_{0j} = -\frac{8\pi G}{c^2}\left\{\frac{\epsilon + p}{c^2}\gamma v_j\right\} + O(\partial\theta)^2$$
$$\frac{1}{2}\Box_g\theta_{jk} - \frac{\epsilon - p}{2c^2}\frac{8\pi G}{c^2}\theta_{jk} = \frac{8\pi G}{c^4}\left\{\frac{\epsilon + p}{c^2}v_j v_k + \frac{\epsilon - p}{2}\delta_{jk}\right\} + O(\partial\theta)^2.$$

Thus, in all these cases, we see that, at first order, $\Delta\theta_{0j} = \Delta\theta_{jk} = 0$ and we can assume that these vanish.

Now to understand the dynamical implication of this, we must compute the Christofel symbols. We find that[8]

$$\Gamma^0_{\mu\nu} = \frac{1}{2}\left[m^{0\omega} - \theta^{0\omega}\right]\left\{\partial_\mu\theta_{\omega\nu} + \partial_\nu\theta_{\omega\mu} - \partial_\omega\theta_{\mu\nu}\right\} = -\frac{1}{2c^2}\left\{\partial_\mu\theta_{\omega\nu} + \partial_\nu\theta_{\omega\mu} - \partial_\omega\theta_{\mu\nu}\right\} + O(|\theta^{0\cdot}|\cdot|\partial\theta|)$$
$$\Gamma^k_{\mu\nu} = \frac{1}{2}\left[m^{k\omega} - \theta^{k\omega}\right]\left\{\partial_\mu\theta_{\omega\nu} + \partial_\nu\theta_{\omega\mu} - \partial_\omega\theta_{\mu\nu}\right\} = \frac{1}{2}\left\{\partial_\mu\theta_{k\nu} + \partial_\nu\theta_{k\mu} - \partial_k\theta_{\mu\nu}\right\} + O(|\theta|\cdot|\partial\theta|),$$

and given our assumptions on θ, we see that the only nonzero coefficients at first order are

$$\Gamma^k_{00} = -\frac{1}{2}\partial_k\theta_{00}. \tag{1.17}$$

Now, the general relativistic version of the equations of motion (1.7) and (1.9) is

$$\nabla_\nu\left[nu^\nu\right] = 0, \qquad \frac{nu^\nu}{c^2}\nabla_\nu\left[hu^\mu\right] + g^{\mu\nu}\nabla_\nu p = 0,$$

which, in our situation, become to first order

$$\partial_t\left[\gamma n\right] + \partial_k\left[n\gamma v^k\right] = 0$$
$$\partial_t[\frac{h}{c^2}\gamma] + v^k\partial_k\left[\frac{h}{c^2}\gamma\right] - \frac{1}{n\gamma}\left[\frac{1}{c^2} + \theta^{00}\right]\partial_t p = 0$$
$$\partial_t\left[\frac{h}{c^2}\gamma v^j\right] + \Gamma^j_{00}\frac{h}{c^2}\gamma + v^k\partial_k\left[\frac{h}{c^2}\gamma v^j\right] + \frac{1}{n\gamma}\partial^j p = 0$$

and using (1.16) and (1.17) and neglecting all the corrections of order c^{-2} (in particular the γ's), we finally obtain

$$\partial_t n + \partial_k\left[nv^k\right] = 0$$
$$nm\left[\partial_t v^j + v^k\partial_k v^j\right] + \partial^j p = \frac{nm}{2}\partial^j\theta_{00}$$
$$-\Delta\theta_{00} = 8\pi G nm$$

and we recognize the Newtonian equations for a perfect gravitating fluid once we set G to be Newton's gravitational constant.

2. Charged fluid

We now introduce an electromagnetic field.

2.1. The Maxwell equations.

[8]From now on, we raise and lower symbols using the Minkowski metric. This is exact at first order since $|\theta| \ll |m|$.

2.1.1. *Electromagnetic field.* An electromagnetic field $F = \{F^{\mu\nu}\}_{0\leq\mu,\nu\leq 3} \in \mathcal{T}^2(M)$ is a skew-symmetric 2-tensor $F^{\mu\nu} = -F^{\nu\mu}$. The Maxwell equations express how an electromagnetic field varies:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \qquad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \tag{2.1}$$

where J^ν denotes the total relativistic current. This can also be rewritten in a more geometric way as

$$dF = 0, \quad d * F = \frac{4\pi}{c} * J, \qquad F = F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

These equations imply the conservation of charge

$$\partial_\nu J^\nu = 0. \tag{2.2}$$

This field has an energy-momentum tensor:

$$\mathcal{E}^{\mu\nu} = -(4\pi)^{-1} \left[F^{\mu\alpha} F^{\beta\nu} g_{\alpha\beta} + \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g^{\mu\nu} \right] \in \mathcal{T}^2(M).$$

in particular, one may verify that $\mathcal{E}$ is a positive tensor.

The classical analogues are defined for the observer v as

$$E^\nu = -ecF^{\mu\nu} v_\mu, \quad B^\alpha = \frac{e}{2c} \epsilon^{\alpha\beta\gamma\delta} v_\beta F_{\gamma\delta}.$$

where e denotes the charge of an electron.

2.1.2. *Electromagnetic potential.* Since the second equation (2.1) can be rewritten as $dF = 0$ and M is simply connected, we see that there exists a 1-form A such that $F = dA$. In other words,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.3}$$

We see from this that A is only determined up to a choice of gauge: if $A' = A + d\chi$, then A' also gives F from the above relation.

For a vector potential A, the first Maxwell equation then gives that

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \left[\partial_\mu A^\mu \right] = \frac{4\pi}{c} J^\nu. \tag{2.4}$$

In particular, given a solution A of (2.4), the electromagnetic field defined by (2.3) automatically satisfies Maxwell's equations.

If we choose the *Lorentz gauge* where $\partial_\mu A^\mu = 0$, using (2.2), we see that (2.4) reduces to

$$\Box A^\nu = -\frac{4\pi}{c} J^\nu, \qquad \partial_\mu A^\mu_{|t=0} = 0. \tag{2.5}$$

2.1.3. *Electromagnetic field in vacuum.* In vacuum, we have no motion of charge and therefore $J \equiv 0$. In this case, we only have to determine the electromagnetic field which satisfies

$$\partial_\mu F^{\mu\nu} = 0 \qquad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \qquad (dF = 0, \ d * F = 0).$$

In particular, we see from (2.5) that Maxwell's equation reduce to

$$\Box F = 0,$$

together with some constraints on the initial data.

2.1.4. *Lorentz force.* The Lorentz force $\mathcal{F}_L$ exerted by the electromagnetic field F onto a particle of charge q moving with 4-velocity u is given by

$$\mathcal{F}_L^\mu = \frac{q}{c} u_\alpha F^{\mu\alpha}.$$

Therefore, naturally, the Lorentz force applied to a fluid with density n, charge-per particle q and 4-velocity u is given by

$$\mathcal{F}_L^\mu = \frac{nq}{c} u_\alpha F^{\mu\alpha}. \tag{2.6}$$

2.2. **One charged fluid in vacuum.**

2.2.1. *Dynamical equations.* We now consider the case of just *one* charged fluid in vacuum, with density of charge $\rho = qn$ for some $q > 0$. In this case, the total electric current is given by

$$J^\nu = qnu^\nu. \tag{2.7}$$

The dynamical equations are then given by the Maxwell equation, the continuity of charge and the conservation of the total stress-energy (or Bianchi identity): (2.1), (2.7), (1.7) and (1.2). The latter reads

$$0 = \partial_\nu \left[T^{\mu\nu} + \mathcal{E}^{\mu\nu}\right] = \frac{nu^\nu}{c^2}\partial_\nu \left[hu^\mu\right] + g^{\mu\nu}\partial_\nu p - \frac{1}{c} J_\alpha F^{\mu\alpha} = \frac{nu^\nu}{c^2}\partial_\nu \left[hu^\mu\right] + g^{\mu\nu}\partial_\nu p - \frac{qn}{c} u_\alpha F^{\mu\alpha}. \tag{2.8}$$

Since F is skew-symmetric, when we project in the direction of u, we obtain (1.10) and thus the discussion in sub subsection 1.5.1 applies equally well. In particular, fluids which are initially isentropic remain so and are in fact barotropic.

2.2.2. *Generalized Vorticity.* We define the *generalized vorticity* as

$$c^2\omega_{\alpha\beta} = \partial_\alpha(hu_\beta) - \partial_\beta(hu_\alpha) + qcF_{\alpha\beta}.$$

This is again transported by the flow in the sense of (1.14). Indeed, we may simply compute

$$\begin{aligned}
c^2u^\nu\partial_\nu\omega_{\alpha\beta} &= \partial_\alpha(u^\nu\partial_\nu(hu_\beta)) - \partial_\nu(hu_\beta)\partial_\alpha u^\nu - \partial_\beta(u^\nu\partial_\nu(hu_\alpha)) + \partial_\nu(hu_\alpha)\partial_\beta u^\nu + qcu^\nu\partial_\nu F_{\alpha\beta}\\
&= -\partial_\alpha(\frac{c^2}{n}\partial_\beta p - cqu_\theta F_{\beta\gamma}g^{\gamma\theta}) + \partial_\beta(\frac{c^2}{n}\partial_\alpha p - cqu_\theta F_{\alpha\gamma}g^{\gamma\theta}) - qcu^\nu(\partial_\alpha F_{\beta\nu} + \partial_\beta F_{\nu\alpha})\\
&\quad - (\partial_\alpha u^\nu)c^2\omega_{\nu\beta} - (\partial_\alpha u^\nu)\partial_\beta(hu_\nu) + qc(\partial_\alpha u^\nu)F_{\nu\beta}\\
&\quad + (\partial_\beta u^\nu)c^2\omega_{\nu\alpha} + (\partial_\beta u^\nu)\partial_\alpha(hu_\nu) - qc(\partial_\beta u^\nu)F_{\nu\alpha}\\
&= qc\{\partial_\alpha(u^\theta F_{\beta\theta}) - \partial_\beta(u^\theta F_{\alpha\theta}) - u^\nu\partial_\alpha F_{\beta\nu} - u^\nu\partial_\beta F_{\nu\alpha} + (\partial_\alpha u^\nu)F_{\nu\beta} - (\partial_\beta u^\nu)F_{\nu\alpha}\}\\
&\quad - (\partial_\alpha u^\nu)c^2\omega_{\nu\beta} + (\partial_\beta u^\nu)c^2\omega_{\nu\alpha} - \{(\partial_\alpha u^\nu)\partial_\beta(hu_\nu) - (\partial_\beta u^\nu)\partial_\alpha(hu_\nu)\}\\
&= -c^2(\partial_\alpha u^\nu)\omega_{\nu\beta} + c^2(\partial_\beta u^\nu)\omega_{\nu\alpha},
\end{aligned}$$

and hence as long as the solution is smooth, irrotational initial data lead to solutions which remain irrotational.

2.3. **The relativistic Euler-Maxwell equation for electrons.** The Euler-Maxwell equation for electrons is the variant of the previous case, where we also assume the presence of a second fluid which remains at rest and which has charge opposite to the fluid under consideration.

We assume the presence of a background fluid with $n \equiv n_0$, $u \equiv \partial_t$ and charge e. Since we assume that this fluid is *not moving*, we cannot incorporate its stress-energy tensor in the Bianchi identity (1.2), which we then have to abandon.

The dynamical equations are then given by the Maxwell equations (2.1), where the total electric current is now

$$J^\nu = e\left[n_0\delta_0^\nu - nu^\nu\right],$$

the conservation of particle (1.7) (or the conservation of charge (2.2)), but we need to find a replacement for (1.2). This is similar to the case of loosing a conservation law, in which case, we can always go back to Newton's law. In special relativity, there are two possible replacements for Newton's law: either a law prescribing the acceleration as in (1.12), or a law prescribing the variation of momentum π (see (1.13)) as in (1.9) or (2.8). The correct one seems the balance of momentum: "the variation of momentum equals the sum of the forces exerted upon the fluid". In our case, we only have the Lorentz force (2.6) and this gives

$$nu^\nu\partial_\nu\pi^\mu + g^{\mu\nu}\partial_\nu p = \frac{en}{c}u_\alpha F^{\mu\alpha}, \quad \pi^\nu = c^{-2}hu^\nu. \tag{2.9}$$

Note that since we do not describe a complete system (e.g. we are neglecting the forces that the electron fluid exerts upon the uniform background), we do not recover the fact that the stress-energy-tensor is divergence free (1.2). However, we retain a partial equality which is sufficient to give us conservation of the physical energy:

$$\partial_\nu\left[T^{0\nu} + \mathcal{E}^{0\nu}\right] = 0.$$

2.3.1. *Lorentz covariance.* Consider a Lorentz-transformation L, i.e. a (fixed) 2-tensor L satisfying $L_{\alpha\beta}L^{\alpha\gamma} = \delta_\beta^\gamma$ and define

$$(X')^\alpha = L^{\alpha\beta}X_\beta, \quad n'(X') = n(X), \quad (u')^\alpha(X') = L^{\alpha\beta}u_\beta(X), \quad (F')^{\alpha\beta}(X') = L^{\alpha\gamma}L^{\beta\delta}F_{\gamma\delta}(X),$$
$$(J')^\alpha(X') = L^{\alpha\beta}J_\beta(X)$$

Then, we see that (n,u,J,F) satisfy (2.1)-(1.7)-(1.2) if and only if (n',u',J',F') does.

2.3.2. *Perturbations of a constant equilibrium.* We remark that the same formal computations as in Subsection 2.2.2 still hold. Therefore it makes sense to consider irrotational fluids. In addition, the following "physical" explanation is sometimes given to justify the study of irrotational initial data:

In the absence of a mechanism to create vorticity (such as the presence of boundary) and when the (generalized) vorticity satisfies a transport law, if one starts with a state which is irrotational, any "physical" perturbation of the system will not destroy this vanishing of the generalized vorticity; in other words, one cannot perturb the velocity *and* the magnetic field independently in a "physical" fashion.

In any event, regardless of the relevance of the previous paragraph, considering an irrotational perturbation greatly simplifies the system and is certainly relevant to applications.

We now consider the perturbation of the following equilibrium:

$$n \equiv n_0, \quad u \equiv \partial_t, \quad F \equiv 0.$$

We assume that the data are isentropic and irrotational. We write the fluid 4-vector as

$$u^\nu = (\gamma_e, v^1, v^2, v^3), \qquad u_\nu = (-c^2\gamma_e, v^1, v^2, v^3), \quad u^\mu u_\mu = -c^2, \quad \gamma_e = \left[1 + c^{-2}|v|^2\right]^{\frac{1}{2}} \tag{2.10}$$

and we consider the unknowns[9]

$$\mu_e^j = c^{-2}hu^j = c^{-2}hv^j, \quad E^j = ecF^{j0}. \tag{2.11}$$

All the other unknowns can be recovered from the formula

$$ec^{-1}F^{jk} = \partial_k\mu_e^j - \partial_j\mu_e^k,$$

the first equation in (2.1)

$$n\gamma_e = n_0 - \frac{1}{4\pi e^2}\partial_j E^j,$$

and from the fact that the mapping

$$D : (n,v) \mapsto (n\gamma_e, \mu_e) = (n\sqrt{1 + c^{-2}|v|^2}, c^{-2}h(n)v)$$

[9]This choice of unknown is motivated from the choice of unknowns in the non relativistic case [5]. Another way of seeing this is that since u^ν is a small variation of ∂_t and since only the equations of motion orthogonal to u^μ will be relevant, it should suffice to look at the projection of the equations of motion onto $\partial_t^\perp$ and to recover the full dynamics of u by imposing (2.10).

is invertible in an L^∞-neighborhood of $v \equiv 0$ and $n \equiv n_0$, where $h \simeq h_0 := h(n_0)$. This latter point is easily seen from the Jacobian matrix

$$\nabla D = \begin{pmatrix} \gamma_e & \gamma_e^{-1} c^{-2} n v^T \\ c^{-2} h'(n) v & c^{-2} h(n) I_3 \end{pmatrix},$$

which also in particular implies that

$$\partial_j h = -\frac{h'(n_0)}{4\pi e^2} \partial_j \partial_k E^k + h.o.t. \tag{2.12}$$

The dynamical equations then reduce to the following (from (1.9) and the first equation of (2.1))

$$\begin{aligned} \partial_t \mu_e^j + E^j + \frac{1}{\gamma_e} \partial_j h + \frac{c^2}{h\gamma_e} \partial_j \frac{|\mu_e|^2}{2} = 0 \\ \partial_t E^j - c^2 (\mathrm{curl}\,\mathrm{curl}(\mu_e))^j - 4\pi e^2 c^2 \frac{n}{h} \mu_e^j = 0, \end{aligned} \tag{2.13}$$

where

$$(\mathrm{curl}(v))^i := \in^{ijk} \partial_j v_k.$$

We can now choose scales appropriately so as to minimize the number of parameters in the linear system. Define[10]

$$\begin{aligned} & h_0 = h(n_0), \quad p_0' = p'(n_0), \qquad T = \frac{p_0'}{h_0} \approx \left(\frac{c_s}{c}\right)^2, \quad \lambda = \sqrt{\frac{4\pi e^2 n_0}{h_0}} \approx c\omega_e, \quad \beta = \lambda c \\ & \mu_e(x,t) = \widetilde{\mu}(\lambda x, \beta t), \quad E(x,t) = \beta \widetilde{E}(\lambda x, \beta t), \\ & n = n_0(1 + \widetilde{n}(\lambda x, \beta t)), \quad h_0 \widetilde{h} = h(\widetilde{n}), \quad \widetilde{\gamma} = \sqrt{1 + (c/h_0)^2 \widetilde{h}^{-2} |\widetilde{\mu}|^2} \end{aligned}$$

and introduce

$$Q = |\nabla|^{-1} \mathrm{curl}, \quad P = -\nabla(-\Delta)^{-1} \mathrm{div}, \quad P^2 + Q^2 = Id, \quad PQ = 0, \quad P^2 = P, \quad Q^3 = Q.$$

We can recast (2.13) as

$$\begin{aligned} \partial_t \widetilde{\mu} + \widetilde{E} - T\Delta P \widetilde{E} = \mathcal{N}_1, \\ \partial_t \widetilde{E} + \Delta Q^2 \widetilde{\mu} - \widetilde{\mu} = \mathcal{N}_2, \end{aligned}$$

where

$$\begin{aligned} -\mathcal{N}_1 &= \frac{c}{h_0} \nabla \frac{|\widetilde{\mu}|^2}{2} + \left\{ T\Delta P \widetilde{E} + \frac{h_0}{\widetilde{\gamma}} \nabla \widetilde{h} \right\} + \frac{c}{h_0} \left\{ \frac{1}{\widetilde{h}\widetilde{\gamma}} - 1 \right\} \nabla \frac{|\widetilde{\mu}|^2}{2} \\ -\mathcal{N}_2 &= \left\{ 1 - \frac{1 + \widetilde{n}}{\widetilde{h}} \right\} \widetilde{\mu}^j. \end{aligned}$$

We define

$$\Lambda_e^2 := 1 - T\Delta, \qquad \Lambda_b^2 := 1 - \Delta$$

and we introduce the dispersive unknowns

$$\begin{aligned} U_e := P\widetilde{\mu} - i\Lambda_e P \widetilde{E} \\ U_b := Q\widetilde{\mu} - i\Lambda_b^{-1} Q \widetilde{E} \end{aligned} \tag{2.14}$$

[10]Here ω_e denotes the (nonrelativist) electron plasma frequency and c_s denotes the (non relativist) sound velocity.

which satisfy the system

$$\begin{aligned}(\partial_t + i\Lambda_e)\, U_e &= P\mathcal{N}_1 - i\Lambda_e P\mathcal{N}_2\\ (\partial_t + i\Lambda_b)\, U_b &= Q\mathcal{N}_1 - i\Lambda_b^{-1} Q\mathcal{N}_2.\end{aligned} \tag{2.15}$$

This system is now amenable to analysis of quasilinear dispersive equations techniques.

2.4. **The relativistic 2-fluid Euler-Maxwell equation.** We now complete the system described above and let the ion evolve freely. This section is essentially taken from [5].

We thus consider two fluids with two densities n_i and n_e, two velocity fields v_i and v_e (both of which satisfy (1.3)) and an electromagnetic field F. We assume that each ion carries a charge of $+Ze$. We are also given pressure laws p_i and p_e and enthalpies h_i and h_e satisfying (1.8), with M_i, the rest-mass of an ion instead of m_e for p_i, h_i. Thus, our matter fields are described by

$$T_i^{\mu\nu} = n_i h_i \frac{u_i^\mu u_i^\nu}{c^2} + p_i g^{\mu\nu}, \qquad T_e^{\mu\nu} = n_e h_e \frac{u_e^\mu u_e^\nu}{c^2} + p_e g^{\mu\nu}.$$

The Maxwell equations (2.1) remain the same, with the relativistic current now defined as

$$J^\nu = Zen_i u_i^\nu - en_e u_e^\nu \tag{2.16}$$

Both species are independently conserved so that

$$\partial_\nu (n_i u_i^\nu) = 0 = \partial_\nu (n_e u_e^\nu) \tag{2.17}$$

and we have two forms of balance of momentum:

$$\begin{aligned}\frac{n_i u_i^\nu}{c^2}\partial_\nu \left[h_i u_i^\mu\right] + g^{\mu\nu}\partial_\nu p_i &= -Z\frac{en_i}{c}(u_i)_\alpha F^{\mu\alpha}\\ \frac{n_e u_e^\nu}{c^2}\partial_\nu \left[h_e u_e^\mu\right] + g^{\mu\nu}\partial_\nu p_e &= \frac{en_e}{c}(u_e)_\alpha F^{\mu\alpha}.\end{aligned} \tag{2.18}$$

In particular, we recover the fact that the stress-energy tensor is divergence free (1.2):

$$\partial_\nu \left[T_i^{\mu\nu} + T_e^{\mu\nu} + \mathcal{E}^{\mu\nu}\right] = 0.$$

Again, we have two naturally transported (generalized) vorticities:

$$\begin{aligned}\omega^i_{\alpha\beta} &= \partial_\alpha \left[h_i (u_i)_\beta\right] - \partial_\beta \left[h_i (u_i)_\alpha\right] - ZecF_{\alpha\beta},\\ \omega^e_{\alpha\beta} &= \partial_\alpha \left[h_e (u_e)_\beta\right] - \partial_\beta \left[h_e (u_e)_\alpha\right] + ecF_{\alpha\beta},\end{aligned}$$

which satisfy that

$$\begin{aligned}u_i^\nu \partial_\nu \omega^i_{\alpha\beta} &= -(\partial_\alpha u_i^\nu)\omega^i_{\nu\beta} + (\partial_\beta u_i^\nu)\omega^i_{\nu\alpha},\\ u_e^\nu \partial_\nu \omega^e_{\alpha\beta} &= -(\partial_\alpha u_e^\nu)\omega^e_{\nu\beta} + (\partial_\beta u_e^\nu)\omega^e_{\nu\alpha}.\end{aligned} \tag{2.19}$$

We thus see that irrotational flows are well-defined and remain irrotational along the flow.

We can easily see from (1.8) that the component of (2.18) parallel to the fluids under consideration are automatically satisfied. Thus to verify (2.18), it suffices to verify it when $\mu = j$ varies between 1 and 3.

We now define the unknowns

$$\begin{aligned}&\mu_i^j = c^{-2} h_i u_i^j, \quad \mu_e^j = c^{-2} h_e u_e^j,\ 1 \leq j \leq 3,\\ &E^j = ecF^{j0}, \quad 2B^j = -ec^{-1} \in^{jkl} F_{kl}, \quad F^{jk} = -ce^{-1} \in^{jkl} B^l.\end{aligned}$$

Now, we can rewrite our evolution system as

$$
\begin{aligned}
\partial_t(n_i\gamma_i) + c^2\mathrm{div}(\frac{n_i}{h_i}\mu_i) &= 0,\\
\partial_t\mu_i + \frac{1}{\gamma_i}\partial_j h_i - ZE^j + \frac{c^2}{\gamma_i h_i}\partial_j\frac{|\mu_i|^2}{2} &= 0,\\
\partial_t(n_e\gamma_e) + c^2\mathrm{div}(\frac{n_e}{h_e}\mu_e) &= 0,\\
\partial_t\mu_e^j + \frac{1}{\gamma_e}\partial_j h_e + E^j + \frac{c^2}{h_e\gamma_e}\partial_j\frac{|\mu_e|^2}{2} &= 0,\\
\partial_t E - c^2\mathrm{curl}(B) + 4\pi e^2c^2[Z\frac{n_i}{h_i}\mu_i - \frac{n_e}{h_e}\mu_e] &= 0,\\
\partial_t B + \mathrm{curl}(E) &= 0.
\end{aligned}
$$

2.4.1. *Linearization at an equilibrium.* From these equations, we can find the equations satisfied by the deviation from the equilibrium state

$$
n_i \equiv Z^{-1}n_0, \quad n_e \equiv n_0, \quad u_e \equiv u_i \equiv \partial_t, \quad F^{\mu\nu} \equiv 0.
$$

We now set

$$
\begin{aligned}
&H_i = h_i(n_0/Z), \quad n_0P_iZ = p_i'(n_0/Z), \quad H_e = h_e(n_0), \quad n_0P_e = p_e'(n_0),\\
&\beta := \sqrt{\frac{4\pi n_0 Ze^2c^2}{H_i}}, \qquad \lambda := \sqrt{\frac{4\pi e^2}{P_i}}, \qquad \mu := \frac{\sqrt{n_0ZP_iH_i}}{c}
\end{aligned}
$$

and

$$
\varepsilon = \frac{ZH_e}{H_i}, \qquad T = \frac{P_e}{P_i}, \qquad C_b = \frac{H_e}{n_0P_i} \tag{2.20}
$$

and use the rescaling

$$
\begin{aligned}
\gamma_i(x,t) &= \widetilde{\gamma}_i(\lambda x, \beta t), & \gamma_e(x,t) &= \widetilde{\gamma}_e(\lambda x, \beta t),\\
n_i(x,t)\gamma_i(x,t) &= (n_0/Z)[\rho(\lambda x, \beta t) + 1], & n_e(x,t)\gamma_e(x,t) &= n_0[n(\lambda x, \beta t) + 1]\\
\mu_i(x,t) &= \mu u(\lambda x, \beta t), & \mu_e(x,t) &= (\varepsilon\mu/Z)v(\lambda x, \beta t)\\
E(x,t) &= n_0\lambda P_i\widetilde{E}(\lambda x, \beta t), & B(x,t) &= (\lambda\mu/Z)\widetilde{B}(\lambda x, \beta t)\\
h_i(n_i(x,t)) &= H_i\widetilde{h}_i(\widetilde{\rho}(\lambda x, \beta t)), & h_i'(n_i) &= Z^2P_iq_i'(\widetilde{\rho}), \qquad \widetilde{h}_i(0) = 1 = q_i'(0)\\
h_e(n_e(x,t)) &= H_e\widetilde{h}_e(\widetilde{n}(\lambda x, \beta t)), & h_e'(n_e) &= P_eq_e'(\widetilde{n}), \qquad \widetilde{h}_e(0) = 1 = q_e'(0)
\end{aligned}
$$

to obtain the system

$$\begin{aligned}
\partial_t\rho+\mathrm{div}[\frac{1+\rho}{\widetilde{\gamma}_i\widetilde{h}_i}u]&=0,\\
\partial_t u^j-\widetilde{E}^j+\frac{1}{\widetilde{\gamma}_i}\partial_j q_i+\frac{1}{\widetilde{\gamma}_i\widetilde{h}_i}\partial_j\frac{|u|^2}{2}&=0,\\
\partial_t n+\mathrm{div}[\frac{1+n}{\widetilde{\gamma}_e\widetilde{h}_e}v]&=0,\\
\varepsilon\{\partial_t v+\frac{1}{\widetilde{h}_e\widetilde{\gamma}_e}\partial_j\frac{|v|^2}{2}\}+\widetilde{E}^j+\frac{T}{\widetilde{\gamma}_e}\partial_j q_e&=0,\\
\partial_t\widetilde{E}^j-\frac{C_b}{\varepsilon}\mathrm{curl}(\widetilde{B})+[\frac{1+\rho}{\widetilde{\gamma}_i\widetilde{h}_i}u-\frac{1+n}{\widetilde{\gamma}_e\widetilde{h}_e}v]&=0\\
\partial_t\widetilde{B}+\mathrm{curl}(\widetilde{E})&=0
\end{aligned}\tag{2.21}$$

which has a similar structure to the classical Euler-Maxwell system. Indeed, we may Taylor expand to get

$$\begin{aligned}
\frac{1+\rho}{\widetilde{\gamma}_i\widetilde{h}_i}&=1+r_1\rho+g_1,\quad q_i=\rho+r_2\rho^2+r_2'|u|^2+h_2,\\
\frac{1+n}{\widetilde{\gamma}_e\widetilde{h}_e}&=1+r_3n+g_3,\quad q_e=n+r_4n^2+r_4'|v|^2+h_4,
\end{aligned}$$

where r_1, r_2, r_2', r_3, r_4 and r_4' are constants and g_1,g_3 are smooth functions of (ρ,u,n,v) which vanish at the origin $(0,0,0,0)$ together with their gradient, and h_2,h_4 are smooth functions of (ρ,u,n,v) which vanish at the origin $(0,0,0,0)$ together with their first and second derivatives.

We may thus rewrite (2.21) as

$$\begin{aligned}
\partial_t\rho+\mathrm{div}[u]+r_1\mathrm{div}[\rho u]&=-\mathrm{div}[g_1],\\
\partial_t u^j-\widetilde{E}^j+\partial_j\rho+r_2\partial_j(\rho^2)+(r_2'+\frac{1}{2})\partial_j|u|^2&=T_2,\\
\partial_t n+\mathrm{div}[v]+r_3\mathrm{div}[nv]&=-\mathrm{div}[g_3],\\
\partial_t v+\varepsilon^{-1}\widetilde{E}^j+\varepsilon^{-1}T\partial_j n+\varepsilon^{-1}Tr_4\partial_j(n^2)+(\varepsilon^{-1}Tr_r'+\frac{1}{2})\partial_j|v|^2&=T_4,\\
\partial_t\widetilde{B}+\mathrm{curl}(\widetilde{E})&=0,\\
\partial_t\widetilde{E}-\frac{C_b}{\varepsilon}\mathrm{curl}(\widetilde{B})+u-v+[r_1\rho u-r_3nv]&=-g_1u+g_3v,
\end{aligned}\tag{2.22}$$

where

$$\begin{aligned}
T_2&=-\{\widetilde{\gamma}_i^{-1}-1\}\partial_j q_i-\widetilde{\gamma}_i^{-1}\partial_j h_2-((\widetilde{\gamma}_i\widetilde{h}_i)^{-1}-1)/2\cdot\partial_j|u|^2\\
T_4&=-\{\varepsilon^{-1}T\widetilde{\gamma}_e^{-1}-1\}\partial_j q_e-\varepsilon^{-1}\widetilde{\gamma}_e^{-1}\partial_j h_4-((\widetilde{h}_e\widetilde{\gamma}_e)^{-1}-1)/2\cdot\partial_j|v|^2
\end{aligned}$$

are simply smooth cubic (or higher order) terms in (ρ,n,u,v) with no particle structure. We can directly observe that the linearization of (2.22) coincides with the linearization of the classical equation. Therefore we consider the same dispersion relations and we define similarly the dispersive unknowns as:

$$\begin{aligned}
U_i = U_{i+} &:= \frac{1}{2\sqrt{1+R^2}}\big[\varepsilon^{1/2}R|\nabla|^{-1}\Lambda_i n + |\nabla|^{-1}\Lambda_i\rho - i\varepsilon^{1/2}RR_j v^j - iR_j u^j\big],\\
U_e = U_{e+} &:= \frac{1}{2\sqrt{1+R^2}}\big[-\varepsilon^{1/2}|\nabla|^{-1}\Lambda_e n + R|\nabla|^{-1}\Lambda_e\rho + i\varepsilon^{1/2}R_j v^j - iRR_j u^j\big],\\
2U_b = 2U_{b+} &:= \Lambda_b|\nabla|^{-1}Q\tilde{B} - iQ^2\tilde{E}
\end{aligned} \tag{2.23}$$

with inverse transformation given by

$$\begin{aligned}
n &= \frac{-|\nabla|\varepsilon^{-1/2}}{\sqrt{1+R^2}\Lambda_e}(U_e+\overline{U_e}) + \frac{|\nabla|\varepsilon^{-1/2}R}{\sqrt{1+R^2}\Lambda_i}(U_i+\overline{U_i}),\\
\rho &= \frac{|\nabla|R}{\sqrt{1+R^2}\Lambda_e}(U_e+\overline{U_e}) + \frac{|\nabla|}{\sqrt{1+R^2}\Lambda_i}(U_i+\overline{U_i}),\\
v^j &= R_j\left\{\frac{i\varepsilon^{-1/2}}{\sqrt{1+R^2}}(U_e-\overline{U_e}) + \frac{-i\varepsilon^{-1/2}R}{\sqrt{1+R^2}}(U_i-\overline{U_i})\right\} + \frac{2}{\varepsilon}\Lambda_b^{-1}\mathrm{Re}(U_b^j),\\
u^j &= R_j\left\{\frac{-iR}{\sqrt{1+R^2}}(U_e-\overline{U_e}) + \frac{-i}{\sqrt{1+R^2}}(U_i-\overline{U_i})\right\} - 2\Lambda_b^{-1}\mathrm{Re}(U_b^j).
\end{aligned} \tag{2.24}$$

We also define $U_{\sigma-} = \overline{U_\sigma}$, $\sigma\in\{i,e,b\}$. Above, we have used the operators:

$$\begin{aligned}
\Lambda_i &:= \varepsilon^{-1/2}\sqrt{\frac{(1+\varepsilon)-(T+\varepsilon)\Delta - \sqrt{((1-\varepsilon)-(T-\varepsilon)\Delta)^2+4\varepsilon}}{2}},\\
\Lambda_e &:= \varepsilon^{-1/2}\sqrt{\frac{(1+\varepsilon)-(T+\varepsilon)\Delta + \sqrt{((1-\varepsilon)-(T-\varepsilon)\Delta)^2+4\varepsilon}}{2}},\\
\Lambda_b &:= \varepsilon^{-1/2}\sqrt{1+\varepsilon-C_b\Delta},\\
H_1 &:= \sqrt{1-\Delta},\qquad H_\varepsilon := \varepsilon^{-1/2}\sqrt{1-T\Delta},\\
R &:= \sqrt{\frac{\Lambda_e^2-H_\varepsilon^2}{H_\varepsilon^2-\Lambda_i^2}},\qquad R_\alpha := |\nabla|^{-1}\partial_\alpha,\qquad Q_{\alpha\beta}(\xi) := |\nabla|^{-1}\in_{\alpha\gamma\beta}\partial_\gamma.
\end{aligned} \tag{2.25}$$

Using these formulas (and in particular the fact that $\partial_t n$ and $\partial_t\rho$ are exact spatial derivatives so as to counteract the singular relation at 0 frequency in the definition of U_e and to keep the derivative structure in the quadratic part of the nonlinearity $\mathcal{N}_i$), one quickly sees that (U_i,U_e,U_b) satisfy

$$(\partial_t+i\Lambda_i)U_i = \mathcal{N}_i,\qquad (\partial_t+i\Lambda_e)U_e = \mathcal{N}_e,\qquad (\partial_t+i\Lambda_b)U_{b,\alpha} = \mathcal{N}_{b,\alpha}, \tag{2.26}$$

where the quadratic nonlinear terms are of the form

$$\begin{aligned}
&\mathcal{F}(\mathcal{N}_\sigma)(\xi,t) = c\sum_{\mu,\nu\in\mathcal{I}_0}\int_{\mathbb{R}^3} m_{\sigma;\mu,\nu}(\xi,\eta)\widehat{U_\mu}(\xi-\eta,t)\widehat{U_\nu}(\eta,t)\,d\eta,\\
&\sigma\in\{i,e,b\},\qquad \mathcal{I}_0 := \{e+,e-,i+,i-,b+1,b+2,b+3,b-1,b-2,b-3\},
\end{aligned} \tag{2.27}$$

where the multipliers $m_{e;\mu,\nu}(\xi,\eta)$ and $m_{b,\alpha;\mu,\nu}(\xi,\eta)$, $\alpha\in\{1,2,3\}$, can be written as finite sums of functions of the form

$$(1+|\xi|^2)^{1/2}\cdot m(\xi,\eta),\qquad m\in\mathcal{M} \tag{2.28}$$

and the multipliers $m_{i;\mu,\nu}(\xi,\eta)$ can be written as finite sums of functions of the form

$$|\xi|\cdot m(\xi,\eta),\qquad m\in\mathcal{M}, \tag{2.29}$$

where $\mathcal{M}$ denotes the set of functions of the form $m(\xi,\eta) = q_1(\xi)q_2(\xi-\eta)q_3(\eta)$ where q_i is a nice symbol.

3. Basic formulas and tools

3.1. **Some geometrical formulas.** We recall the definition of the relevant geometrical objects, given a manifold (M,g) with a Levi-Civita connection:

$$\begin{aligned}\Gamma^\gamma_{\alpha\beta} &= \frac{1}{2}g^{\gamma\theta}\left\{\partial_\alpha g_{\theta\beta} + \partial_\beta g_{\theta\alpha} - \partial_\theta g_{\alpha\beta}\right\}\\ \mathrm{R}^\mu{}_{\alpha\beta\gamma} &= \partial_\beta\Gamma^\mu_{\alpha\gamma} - \partial_\gamma\Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\gamma\alpha} - \Gamma^\mu_{\sigma\gamma}\Gamma^\sigma_{\alpha\beta}\\ \mathrm{Ric}_{\mu\nu} &= \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\alpha\mu} = \mathrm{R}^\alpha{}_{\mu\alpha\nu}\\ R &= g^{\alpha\beta}\mathrm{Ric}_{\alpha\beta}.\end{aligned}$$

We claim that, in Lorentz coordinates [11],

$$2(\mathrm{Ric}_g)_{\mu\nu} = \Box_g g_{\mu\nu} + O(\partial h)^2. \tag{3.1}$$

Indeed, in Lorentz-coordinates, we have that

$$2g^{\alpha\beta}\partial_\alpha g_{\theta\beta} = g^{\alpha\beta}\partial_\theta g_{\alpha\beta} = 0 \Leftrightarrow \partial_\beta\left\{g^{\theta\beta}\sqrt{|\det g|}\right\} = 0 \Leftrightarrow 2\partial_\sigma g^{\sigma\nu} = g_{\alpha\beta}g^{\mu\nu}\partial_\mu g^{\alpha\beta}. \tag{3.2}$$

This gives that

$$2\mathrm{Ric}_{\mu\nu} = \partial_\sigma\left\{g^{\sigma\theta}[\partial_\mu g_{\theta\nu} + \partial_\nu g_{\theta\mu} - \partial_\theta g_{\mu\nu}]\right\} - \partial_\nu\left\{g^{\alpha\theta}\partial_\mu g_{\theta\alpha}\right\} + \Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\alpha\mu}$$

Keeping only the terms linear in the metric, we get

$$2(\mathrm{Ric}_g)_{\mu\nu} = -\partial^\theta\partial_\theta g_{\mu\nu} + \left\{\partial_\mu[g^{\sigma\theta}\partial_\sigma g_{\theta\nu}] + \partial_\nu[g^{\theta\sigma}\partial_\sigma g_{\theta\mu}] - \partial_\nu[g^{\alpha\theta}\partial_\mu g_{\theta\alpha}]\right\} + O(\partial g)^2.$$

In view of (3.2), the term inside the bracket only produces commutator terms which are $O(\partial g)^2$ and we therefore obtain (3.1).

As an aside that we will not use, we may remark that in Lorentz coordinates, we have that, for any function ϕ,

$$\Box_g\phi = g^{\alpha\beta}\partial_{\alpha\beta}\phi.$$

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[11] Below $\Box_g g_{\mu\nu}$ denotes of course the D'Alembertian of the *function* $g_{\mu\nu}$. By metric compatibility, the D'Alembertian of the metric vanishes identically.