SEMILINEAR DISPERSIVE EQUATIONS

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1. Examples of normal form transformations

Normal form transformations correspond to a change of unknown $u \to v := u + O(u^2)$ which allows to remove some parts of the nonlinearity (at least up to terms of higher homogeneity)

$$\mathcal{L}u = O(u^2) = N_{res}(u) + N_{nres}(u) \qquad \to \qquad \mathcal{L}(v) = N_{res}(u) + O(u^3).$$

It turns out that such transformations can often be found for oscillatory ODEs and, after passing to Fourier space, for dispersive equations (see e.g. [10]).

1.1. **Null form for the wave equation.** A simple example of normal form transformation occurs for the null nonlinear wave equation[1]

$$\Box u = Q_0(u,u) = (\partial_t u)^2 - |\nabla_x u|^2$$

In this case, we observe that the simple change of unknowns $u \mapsto v := u + u^2/2$ gives

$$\Box v = \Box(u + u^2/2) = u\Box u = u\left((\partial_t u)^2 - |\nabla_x u|^2\right)$$

and this conjugates the original quadratic equation for u into a cubic equation for v. We can also note that this is only the first term of a complete normal form

$$\Box(e^u - 1) = e^u\left(\Box u - (\partial_t u)^2 + |\nabla_x u|^2\right) = 0,$$

so that $w = e^u - 1$ evolves completely linearly.

1.2. **KdV on $\mathbb{T}$ in L^2.** This follows essentially [1]. We consider the KdV equation on the Torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$(\partial_t + \partial_{xxx})\, u = \partial_x(u^2), \qquad u(x,t) : \mathbb{T}\times\mathbb{R} \to \mathbb{R}. \tag{1.1}$$

Conjugating by the free flow and taking the Fourier transform, we can write

$$u(x,t) = \sum_{k\in\mathbb{Z}} a_k(t) e^{i\left[kx + k^3 t\right]}, \qquad a_k(t) = \overline{a}_{-k}(t)$$

and we obtain

$$(\partial_t + \partial_{xxx})\, u = \sum_{k\in\mathbb{Z}} \partial_t a_k(t) e^{i\left[kx+k^3t\right]},$$
$$\partial_x(u^2) = i(p+q)\sum_{p,q\in\mathbb{Z}} a_p(t)a_q(t) e^{i\left[(p+q)x + (p^3+q^3)t\right]}$$

[1]here we denote the D'Alembertian by $\Box u = \left(-\partial_t^2 + \Delta\right) u = m^{\alpha\beta}\partial_\alpha\partial_\beta u$.

and matching the oscillatory phase[2], we obtain the ODE system

$$\partial_t a_k(t) = ik \sum_{\substack{p,q\in\mathbb{Z},\\ p+q=k}} a_p(t)a_q(t)e^{i\left[p^3+q^3-k^3\right]t}, \qquad k\in\mathbb{Z}. \tag{1.2}$$

We directly remark that $\partial_t a_0 = 0$, and in fact, up to replacing

$$u \to v(x,t) := u(x-2ct,t), \qquad c := \frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} u_0 dx,$$

we may assume that $a_0(t)\equiv 0$. In addition, if $p+q=k$, we can decompose

$$p^3+q^3-k^3 = p^3+q^3-(p+q)^3 = -3pq(p+q)$$

and we can partially integrate (1.2) by integrating the fast oscillation:

$$\begin{aligned}\partial_t a_k(t) &= -ik \sum_{\substack{p,q\in\mathbb{Z},\\ p+q=k}} a_p(t)a_q(t)\partial_t\left\{\frac{e^{i\left[p^3+q^3-k^3\right]t}}{i3kpq}\right\},\\ &= -\partial_t\left\{\frac{1}{3}\sum_{\substack{p,q\in\mathbb{Z},\\ p+q=k}} \frac{a_p(t)}{p}\cdot\frac{a_q(t)}{q}\cdot e^{i\left[p^3+q^3-k^3\right]t}\right\}\\ &\quad + \frac{2}{3}\sum_{\substack{p,q\in\mathbb{Z},\\ p+q=k}} \frac{a_p(t)}{p}\cdot\frac{\partial_t a_q(t)}{q}\cdot e^{i\left[p^3+q^3-k^3\right]t},\end{aligned} \tag{1.3}$$

and letting

$$A_k(t) := a_k(t) + \frac{1}{3}\sum_{\substack{p,q\in\mathbb{Z},\\ p+q=k}} \frac{a_p(t)}{p}\cdot\frac{a_q(t)}{q}\cdot e^{i\left[p^3+q^3-k^3\right]t}$$

we see that $a_k \mapsto A_k$ is a local change of variable on $\mathfrak{h}^s$, $s\ge 0$ with

$$\|\langle k\rangle(A_k-a_k)\|_{\ell^2(\mathbb{Z})} \lesssim \|a\|^2_{\ell^2(\mathbb{Z})} \tag{1.4}$$

and

$$\partial_t A_k = -\frac{2}{3}i\sum_{\substack{p,q,r\in\mathbb{Z},\\ p+q+r=k}} \frac{a_p(t)}{p}\cdot a_q(t)\cdot a_r(t)\cdot e^{i\left[p^3+q^3+r^3-k^3\right]t} = T(a,a,a) \tag{1.5}$$

and we note that T is bounded in $\ell^1(\mathbb{Z})\subset\mathfrak{h}^1$ so that the regular Cauchy-Lipshitz gives

Theorem 1.1. *Any initial data $u_0\in H^1(\mathbb{T})$ leads to a local solution $u\in C([0,T]:H^1(\mathbb{T}))$ where $T=T(\|u_0\|_{H^1})$. Using the conservation laws, this solution can be extended to a global solution.*

Proof of Theorem 1.1. It follows from (1.2) that

$$M := \|a(t)\|^2_{\ell^2} = \|u_0\|^2_{L^2}$$

is conserved. We see from (1.4) that

$$\|a\|_{\mathfrak{h}^1} \le \|A\|_{\mathfrak{h}^1} + O(M).$$

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[2]or taking the Fourier transaform in x.

In addition, inspection of (1.5) using the algebra property gives that

$$\|T(a,b,c)\|_{\mathfrak{h}^1} \lesssim \|a\|_{\mathfrak{h}^1}\|b\|_{\mathfrak{h}^1}\|c\|_{\mathfrak{h}^1}$$

so that the ODE for A has a local solution on a time interval $[0,T]$ with $T = T(\|a(0)\|_{\mathfrak{h}^1})$. The solution can be extended globally through the conservation of the energy

$$E(u) := \frac{1}{2}\int_{\mathbb{T}} (\partial_x u)^2 dx + \frac{1}{3}\int_{\mathbb{T}} u^3 dx$$

which, together with the conservation of the mass, controls the $\dot{H}^1$ norm through Sobolev embedding:

$$\left|\int_{\mathbb{T}} u^3 dx\right| \le \|u\|_{L^\infty} M(u) \lesssim M(u)^{\frac{3}{2}} + M(u)\|\partial_x u\|_{L^2}.$$

□

In fact, using that

$$p^3 + q^3 + r^3 - k^3 = 3(p+q)(p+r)(q+r)$$

the normal form transformation can be iterated to give $L^2(\mathbb{T})$-LWP[3] but the computations become more cumbersome since some part of the nonlinearity becomes resonant (e.g. when $p+q=0$ - observe that in this case, one can use that $\sum_p a_p a_{-p} = \sum_p a_p \overline{a}_p = \|a\|_{\ell^2}$ is conserved) and cannot be integrated.

2. An example of linear scattering for the nonlinear Schrödinger equation

This is more or less taken from [2, 11]. We will consider the following equation for $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$:

$$(i\partial_t - \Delta)\, u + |u|^2 u = 0, \qquad u(t=0) = u_0 \in H^1(\mathbb{R}^3). \tag{2.1}$$

Solutions are global by a continuation argument thanks to the conservation laws[4]

$$M(u) := \int |u(x,t)|^2 dx = \int |u_0(x)|^2 dx, \qquad E(u) := \int \left\{ |\nabla u|^2 + \frac{1}{2}|u|^4 \right\} dx = E(u_0). \tag{2.2}$$

We propose to prove the following:

Theorem 2.1. *Any solution of (2.1) starting from H^1 initial data leads to a unique global solution $u \in C_t(\mathbb{R} : H^1(\mathbb{R}^3)) \cap L^5_{x,t}(\mathbb{R}^3 \times \mathbb{R})$ that scatters linearly in the sense that there exists $u^\pm \in H^1$ such that*

$$\|u(t) - e^{it\Delta}u^\pm\|_{H^1} \to 0 \text{ as } t \to \pm\infty.$$

In addition, we have propagation of regularity: assuming that $u_0 \in H^s$, $s \ge 1$, then $u \in C_t(\mathbb{R} : H^s)$ and

$$\|u^\pm\|_{H^s} \lesssim C\|u_0\|_{H^s}.$$

[3] see [1] for details

[4] the first can be verified by multiplying the equation by $\overline{u}$, taking the imaginary part and integrating. The second can be obtained by multiplying the equation by $\partial_t \overline{u}$, taking the real part and integrating.

Remark 2.1. (1) *The theorem remains true in arbitrary dimension $d \geq 1$, provided one replaces the nonlinearity $|u|^2 u$ by $|u|^{p-1}u$ for $1 + 4/d \leq p \leq 1 + 4/(d-2)$ ($1 + 4/d \leq p < \infty$ when $d = 1, 2$), although the proof gets considerably more complicated in the endpoint case.*

(2) *The problem is more complicated in the focusing case when $+|u|^2 u$ is replaced by $-|u|^2 u$. In this case, one needs to add at least some solitons, but the general picture is still largely conjectural.*

We will focus on the case $t \to +\infty$. Thanks to the time-reversal symmetry

$$u(x, t) \to \overline{u}(x, -t)$$

the result for the backward case $t \to -\infty$ follows from the forward case. Before we continue with the proof, we can observe that the proof of *weak* scattering is simple (see Lemma 2.1). We also recall that (2.1) is invariant by the scaling

$$u(x, t) \mapsto \lambda u(\lambda x, \lambda^2 t), \qquad u_0(x) \mapsto \lambda u_0(\lambda x) \tag{2.3}$$

which preserves the $\dot{H}^{\frac{1}{2}}$-norm of the initial data. And therefore, for control of global behavior, only scale-invariant norms can be helpful. Fortunately, the conservation laws (2.2) provides us with two off-scale global bounds which can then be used to interpolate with many more off-scale bounds to produce a scale-invariant norm. Starting from (2.2), we already see that

$$\|u\|_{L^\infty \dot{H}^{\frac{1}{2}}} \lesssim \|u\|_{L^\infty_t L^2_x}^{\frac{1}{2}} \|\nabla_x u\|_{L^\infty_t L^2_x}^{\frac{1}{2}} \lesssim \sqrt{M(u)E(u)}$$

is a first scale-invariant quantity. If this norm is small, local theory gives the scattering result easily. When this norm is large, we need to obtain a norm that can be made small for large time. Here we will choose a Strichartz norm $L^p_t L^q_x$ with $p < \infty$ (where functions of compact support in time are dense).

2.1. **Reduction to a quantitative estimate.**

2.1.1. *The Duhamel formula.* If we consider a smooth solution, we may conjugate the equation by $e^{it\Delta}$ and rewrite

$$\partial_t \left\{ e^{it\Delta} u(t) \right\} = i e^{it\Delta} \left\{ |u(t)|^2 u(t) \right\} \tag{2.4}$$

which, upon integration, leads to

$$\begin{aligned} u(t) &= e^{-it\Delta} u_0 + i \int_{s=0}^{t} e^{-i(t-s)\Delta} \left\{ |u(s)|^2 u(s) \right\} ds \\ &= e^{-it\Delta} \left\{ u_0 + i \int_{s=0}^{t} e^{-is\Delta} \left\{ |u(s)|^2 u(s) \right\} ds \right\} \end{aligned}$$

and we see that scattering is more or less equivalent to making sense of the indefinite integral

$$u^+ := u_0 + i \int_{s=0}^{\infty} e^{-is\Delta} \left\{ |u(s)|^2 u(s) \right\} ds. \tag{2.5}$$

Given the conservation laws, this integral converges weakly.

Lemma 2.1. *Let u be a solution to (2.1) with H^1 initial data. This solution converges weakly in the sense that there exists $u^\pm \in H^1$ such that for all $\phi \in C^\infty_c(\mathbb{R}^3)$, there holds that*

$$\langle u(t), e^{-it\Delta}\phi \rangle \to \langle u^\pm, \phi \rangle, \qquad \text{as } t \to \pm\infty. \tag{2.6}$$

Proof of Lemma 2.1. Given $\phi \in C_c^\infty(\mathbb{R}^3)$, we take the inner product and use the linear dispersion to get

$$\begin{aligned} m_\phi(t) := \langle u(t), e^{-it\Delta}\phi\rangle = \langle e^{it\Delta}u(t),\phi\rangle \\ = \langle u_0,\phi\rangle + i\int_{s=0}^t \langle e^{-is\Delta}\left\{|u(s)|^2u(s)\right\},\phi\rangle ds \\ = \langle u_0,\phi\rangle + i\int_{s=0}^t I(s)ds, \qquad I(s) := \langle |u(s)|^2u(s), e^{is\Delta}\phi\rangle \end{aligned}$$

and it suffices to show that $I(s)$ is integrable. But using the dispersion inequality (2.11) and the conservation laws, we see that

$$|I(s)| \lesssim \|e^{is\Delta}\phi\|_{L^\infty}\|u^3(s)\|_{L^1} \lesssim_\phi \langle s\rangle^{-\frac{3}{2}}\|u(s)\|_{L^3}^3 \lesssim_\phi \langle s\rangle^{-\frac{3}{2}}\left[M(u)+E(u)\right]^{\frac{3}{2}}.$$

Besides, we see that

$$\|u^\pm\|_{H^1} \le \liminf \|e^{it\Delta}u(t)\|_{H^1} = \|u(t)\|_{H^1} \le \left[M(u)+E(u)\right]^{\frac{1}{2}}.$$

□

In fact, with an appropriate spacetime bound, we can upgrade this to strong convergence.

Lemma 2.2. *Assume that $u \in L^5_{x,t}(\mathbb{R}^3\times\mathbb{R})$, then u scatters. In particular u scatters if $\|u\|_{\dot H^{\frac{1}{2}}} < \varepsilon$ is small enough.*

Proof of Lemma 2.2. This follows from Strichartz estimates. Up to choosing T large enough, we can assume that

$$\|u\|_{L^5_t([T,\infty):L^5_x)} < \varepsilon$$

and Strichartz estimates give that for $P \in \{Id, \nabla_x\}$, on $I = [T, T^*)$

$$\begin{aligned} \|Pu\|_{S^0(I)} &\lesssim \|Pu(T)\|_{L^2} + \|u^2Pu\|_{L^{\frac{10}{7}}_{x,t}} \\ &\lesssim \|u(T)\|_{H^1} + \|u^2\|_{L^{\frac{5}{2}}_{x,t}}\|Pu\|_{L^{\frac{10}{3}}_{x,t}} \\ &\lesssim \|u(T)\|_{H^1} + \varepsilon^2\|Pu\|_{S^0(I)} \end{aligned}$$

and we can absorb the right-hand side into the left hand side to obtain a uniform bound: $u, \nabla u \in L^{\frac{10}{3}}_{x,t}$, and redoing the computations, we see that

$$\begin{aligned} \|P\int_{s=\sigma}^\tau e^{-is\Delta}\left\{|u(s)|^2u(s)\right\}ds\|_{L^2} &\lesssim \|u^2Pu\|_{L_t^{\frac{10}{7}}([\sigma,\infty):L_x^{\frac{10}{7}})} \\ &\lesssim \|u\|^2_{L^5_t([\sigma,\infty)L^5_x)}\|Pu\|_{L^{\frac{10}{3}}_{x,t}} \end{aligned}$$

and we have a Cauchy sequence as $\sigma \to \infty$.

□

Lemma 2.3. *Let u be a solution to* (2.1) *with initial data $u_0 \in H^1$. Assume that*

$$\|u\|_{L^4_{x,t}(\mathbb{R}^3\times\mathbb{R})} < \infty \tag{2.7}$$

then in fact the conclusion of Theorem 2.1 holds.

Proof of Lemma 2.3. **Step 1**: Boundedness of L^4 implies boundedness of scale invariant norms[5].

Splitting the time axis into $N = O(\varepsilon^{-4}\|u\|^4_{L^4_{x,t}})$ intervals, we may assume that

$$\|u\|_{L^4_{x,t}(\mathbb{R}^3\times I)} \le \varepsilon. \tag{2.8}$$

We first use the conservation laws to obtain by interpolation a scale-invariant norm:

$$\|u\|_{L^6_t(I:L^{\frac{9}{2}}_x)} \le \|u\|^{\frac{1}{3}}_{L^\infty_t L^6_x}\|u\|^{\frac{2}{3}}_{L^4_{x,t}(\mathbb{R}^3\times I)} \lesssim_E \varepsilon^{\frac{2}{3}}.$$

Once we have a scale-invariant quantity, we can use it to control other Strichartz norms: using the admissible pairs $(a,b) = (10/3, 10/3)$ and $(p,q) = (30/11, 90/23)$ in (2.13), we find that, if $I = [t_0, t_1]$,

$$\begin{aligned}
\|u\|_{L^{\frac{10}{3}}_{x,t}(\mathbb{R}^3\times[t_0,t_1])} &\lesssim \|e^{-it\Delta}u(t_0)\|_{L^{\frac{10}{3}}_{x,t}} + \|\int_{t_0}^s e^{-i(t-s)\Delta}\left\{|u|^2u(s)\right\}ds\|_{L^{\frac{10}{3}}_{x,t}(\mathbb{R}^3\times[t_0,t_1])}\\
&\lesssim \|u(t_0)\|_{L^2_x} + \|u^3\|_{L^{\frac{30}{19}}_t([t_0,t_1]:L^{\frac{90}{67}}_x)} \lesssim M(u)^{\frac{1}{2}} + \|u\|^2_{L^6_t([t_0,t_1]:L^{\frac{9}{2}}_x)}\|u\|_{L^{\frac{10}{3}}_{x,t}}\\
&\lesssim M(u)^{\frac{1}{2}} + \varepsilon^{\frac{4}{3}}\|u\|_{L^{\frac{10}{3}}_{x,t}}
\end{aligned}$$

and choosing $\varepsilon > 0$ small enough we obtain that

$$\|u\|_{L^{\frac{10}{3}}_{x,t}} \lesssim M(u)^{\frac{1}{2}}.$$

Similarly, we can obtain an H^1-bound using $(a,b) = (10, 30/13)$ and $(p,q) = (30/17, 90/11)$,

$$\begin{aligned}
\|\nabla_x u\|_{L^{10}_t L^{\frac{30}{13}}_x} &\lesssim \|e^{-it\Delta}\nabla_x u(t_0)\|_{L^{10}_t L^{\frac{30}{13}}_x} + \|\int_{t_0}^s e^{-i(t-s)\Delta}\nabla_x\left\{|u|^2u(s)\right\}ds\|_{L^{10}_t L^{\frac{30}{13}}_x}\\
&\lesssim \|\nabla u(t_0)\|_{L^2} + \|u^2\nabla u\|_{L^{\frac{30}{13}}_t([t_0,t_1]:L^{\frac{90}{79}}_x)} \lesssim E(u)^{\frac{1}{2}} + \|u\|^2_{L^6_t([t_0,t_1]:L^{\frac{9}{2}}_x)}\|\nabla_x u\|_{L^{10}_t L^{\frac{30}{13}}_x}\\
&\lesssim E(u)^{\frac{1}{2}} + \varepsilon^{\frac{4}{3}}\|\nabla_x u\|_{L^{10}_t L^{\frac{30}{13}}_x}.
\end{aligned}$$

Now using Sobolev inequality, we conclude that

$$\|u\|_{L^{10}_{x,t}} \lesssim \|\nabla_x u\|_{L^{10}_t L^{\frac{30}{13}}_x} \lesssim E(u)^{\frac{1}{2}}$$

and therefore, summing over the intervals, we see that

$$\|u\|_{L^p(\mathbb{R}^3\times\mathbb{R})} \lesssim \|u_0\|_{H^1}, \qquad 10/3 \le p \le 10.$$

Step 2: Scattering.
We want to show that

$$v(t) = e^{it\Delta}u(t) = u_0 + i\int_{s=0}^t e^{-is\Delta}\left\{|u(s)|^2u(s)\right\}ds$$

[5]note that $L^4_{x,t}$ scales like the $\dot{H}^{\frac{1}{4}}$ norm (see (2.9)), which is supercritical.

is H^1-Cauchy in t. But using the Strichartz estimates (2.13), we obtain that for $\tau \le \tau'$,

$$\begin{aligned}\|v(\tau') - v(\tau)\|_{L^2} &= \| \int_{\mathbb{R}} e^{-is\Delta} \left\{ |u(s)|^2 u(s) \right\} \mathbb{1}_{[\tau,\tau']} ds \|_{L^2} \\ &\lesssim \|u^3\|_{L^{\frac{10}{7}}_{x,t}(\mathbb{R}^3 \times [\tau,\tau'])} \lesssim \|u^2\|_{L^{\frac{5}{2}}_{x,t}(\mathbb{R}^3 \times [\tau,\infty)} \|u\|_{L^{\frac{10}{3}}_{x,t}}\end{aligned}$$

and similarly,

$$\begin{aligned}\|\nabla_x(v(\tau') - v(\tau))\|_{L^2} &= \| \int_{\mathbb{R}} e^{-is\Delta} \nabla_x \left\{ |u(s)|^2 u(s) \right\} \mathbb{1}_{[\tau,\tau']} ds \|_{L^2} \\ &\lesssim \|u^2 \nabla_x u\|_{L^{\frac{10}{7}}_{x,t}(\mathbb{R}^3 \times [\tau,\tau'])} \lesssim \|u^2\|_{L^{\frac{5}{2}}_{x,t}(\mathbb{R}^3 \times [\tau,\infty))} \|\nabla_x u\|_{L^{\frac{10}{3}}_{x,t}}\end{aligned}$$

which is Cauchy as $\tau \to \infty$.

□

Remark 2.2. *The numerology in the choice of the spacetime norms can be understood from the fact that the linearized equation*

$$\mathcal{L}_u v := (i\partial_t - \Delta)\, v + 2|u|^2 v + u^2 \overline{v}$$

defines a bounded mapping $S^0(I) \to S^0(I)$ when u is bounded in a scale invariant norm $\|u\|_{L^p(I:L^q)} < \infty$ for

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2} - s, \qquad d = 3,\ s = \frac{1}{2}. \tag{2.9}$$

The precise choices are dictated by various applications of Hölder and Sobolev inequalities.

2.2. **Interaction Morawetz estimates.** We recall the energy-momentum tensor for (2.1)

$$\begin{aligned} T_{00} &= \frac{1}{2}|u|^2, \qquad T_{0j} = T_{j0} = -\Im\{\overline{u}\partial_{x^j} u\}, \\ T_{jk} &= 2\Re\left\{\partial_{x^j} u \overline{\partial_{x^k} u}\right\} + \frac{1}{2}\delta_{jk}\left[|u|^4 - \Delta(|u|^2)\right], \end{aligned} \tag{2.10}$$

which satisfies the local conservation law:

$$\partial_t T_{0\alpha} = \partial_k T_{k\alpha}, \qquad \alpha \in \{0,1,2,3\}.$$

In particular, one obtains a conserved measure $\mu(dx) = 2T_{00}dx = |u(x,t)|^2 dx$, and a Virial estimate[6]

$$\begin{aligned} \partial_t \int_{\mathbb{R}^3} \{A_j(x-x_0) T_{j0}(x)\}\, dx &= \int_{\mathbb{R}^3} T_{jk}(x) \partial_k A_j(x-x_0) dx \\ &= 2\int_{\mathbb{R}^3} \Re\left\{\partial_{x^j} u(x) \overline{\partial_{x^k} u(x)}\right\} \partial_k A_j(x-x_0) dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^4 \partial_j A_j(x-x_0) dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^2 \Delta(\partial_j A_j(x-x_0)) dx. \end{aligned}$$

[6] intuitively, this reflects the fact that whereas position and momentum can evolve in arbitrary way, position× momentum has a good chance to be monotonic in time.

This can be used for various choices of vector A, with $A = \nabla a$ for $a(x) = |x|^2/2$ or $a(x) = |x|$ or rescaled version being the most frequent (see also [8, 9] for interesting variants in lower dimensions). Choosing $a(x) = |x|$ and using that

$$A_j = \frac{x_j}{|x|}, \qquad \partial_k A_j = \frac{1}{|x|}\left\{\delta_{jk} - \frac{x_j x_k}{|x|^2}\right\}, \qquad \Delta a = \frac{2}{|x|}, \qquad \Delta^2 a = -8\pi\delta_0$$

gives

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^3} \frac{x-x_0}{|x-x_0|}\Im\left\{\overline{u(x)}\partial_{x^j}u(x)dx\right\}dx\\
=&2\int_{\mathbb{R}^3}\Re\left\{\partial_{x^j}u(x)\overline{\partial_{x^k}u(x)}\right\}\cdot\partial_{x^j}\partial_{x^k}a(x-x_0)dx + \int_{\mathbb{R}^3}\frac{|u(x)|^4}{|x-x_0|}dx + 4\pi|u(x_0)|^2.
\end{aligned}$$

The last two terms of the last line are nonnegative; since a is convex, the first term is the contraction of two nonnegative matrices and is guaranteed to be nonnegative. Integrating in time, we see that

$$\begin{aligned}
\|u\|_{L^\infty_t L^2_x}\|\nabla_x u\|_{L^\infty_t L^2_x} &\geq \left[\int_{\mathbb{R}^3}\frac{x-x_0}{|x-x_0|}\Im\left\{\overline{u(x)}\partial_{x^j}u(x)dx\right\}dx\right]_{t_0}^{t_1}\\
&\geq \int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|u(x)|^4}{|x-x_0|}dxdt + 4\pi\int_{\mathbb{R}}|u(x_0)|^2dt.
\end{aligned}$$

This inequality holds for all choices of x_0. It can be used by averaging it over the conserved measure $\mu = |u|^2dx$. This gives

$$\begin{aligned}
&\partial_t \iint_{\mathbb{R}^3}\{T_{00}(y)\partial_j a(x-y)T_{j0}(x)\}dxdy\\
=&\iint_{\mathbb{R}^3}\{T_{00}(y)T_{jk}(x) + T_{j0}(x)T_{k0}(y)\}\partial_k\partial_j a(x-y)dxdy\\
=&\iint_{\mathbb{R}^3}\left[|u(y)|^2\Re\left\{\partial_{x^j}u(x)\overline{\partial_{x^k}u(x)}\right\} + T_{0j}(x)T_{0k}(y)\right]\partial_j\partial_k a(x-y)dxdy\\
&+\frac{1}{4}\iint_{\mathbb{R}^3}|u(y)|^2|u(x)|^4\Delta a(x-y)dxdy - \frac{1}{4}\iint_{\mathbb{R}^3}|u(y)|^2|u(x)|^2\Delta^2 a(x-y)dxdy.
\end{aligned}$$

Once again, we can safely ignore the first term on the RHS. In addition, we see that

$$\begin{aligned}
I(t) :=& \iint_{\mathbb{R}^3}\left\{T_{00}(y)\frac{(x-y)_j}{|x-y|}T_{j0}(x)\right\}dxdy\\
=& \frac{1}{2}\iint_{\mathbb{R}^3}\left\{|u|^2(y)\frac{(x-y)_j}{|x-y|}\Im\left\{\overline{u}(x)\partial_{x^j}u(x)\right\}\right\}dxdy \lesssim \|u\|^3_{L^\infty_t L^2_x}\|\nabla_x u\|_{L^\infty_t L^2}
\end{aligned}$$

and integrating in time, we obtain that

$$\int_{\mathbb{R}}\iint_{\mathbb{R}^3}\frac{|u(x)|^4|u(y)|^2}{|x-y|}dxdydt + 4\pi\int_{\mathbb{R}}\int_{\mathbb{R}^3}|u(x)|^4dxdt \leq 2M(u)^{\frac{3}{2}}E(u)^{\frac{1}{2}}$$

which gives (2.7).

Remark 2.3. (1) *In the radial setting, one could combine the original Morawetz inequality and the Strauss inequality*

$$r^{\frac{1}{2}}|u(r)| \lesssim \|\partial_r u\|_{L^2}$$

for radial functions to get other integral quantities, e.g.

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3}|u(x,t)|^6dxdt \leq \||x|^{\frac{1}{2}}u\|^2_{L^\infty_{x,t}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\frac{|u(x,t)|^4}{|x|}dxdt \lesssim \|u\|_{L^\infty_tL^2_x}\|\nabla_xu\|^3_{L^\infty_tL^2_x}.$$

(2) *Using Littlewood-Paley analysis, one may improve the regularity requirements to more balanced estimates.*

2.3. **Toolbox: linear estimates.** The basic estimate is the dispersion inequality

$$\|e^{-it\Delta}f\|_{L^\infty_x} \lesssim (4\pi t)^{-\frac{3}{2}}\|f\|_{L^1_x} \tag{2.11}$$

which follows from stationary phase, or the representation of the Schrödinger semigroup. It can be interpolated with the fact that $e^{-it\Delta}$ is an L^2 isometry to obtain

$$\|e^{-it\Delta}f\|_{L^p_x} \lesssim (4\pi t)^{-\frac{3}{2}(1-\frac{2}{p})}\|f\|_{L^{p'}_x}, \qquad 2\leq p\leq\infty. \tag{2.12}$$

From (2.11) and the fact that $e^{it\Delta}$ is an L^2 isometry, we can deduce the Strichartz estimates

$$\begin{aligned}\|e^{-it\Delta}f\|_{L^p_tL^q_x} &\lesssim \|f\|_{L^2_x}, \qquad \frac{2}{p}+\frac{3}{q}=\frac{3}{2}, \quad p\geq 2,\\ \|\int e^{is\Delta}h(s)ds\|_{L^2_x} &\lesssim \|h\|_{L^{p'}_tL^{q'}_x},\\ \|\int_{s=0}^t e^{-i(t-s)\Delta}h(s)ds\|_{L^a_tL^b_x} &\lesssim \|h\|_{L^{p'}_tL^{q'}_x}\end{aligned} \tag{2.13}$$

where (a,b) satisfies the same requirements as (p,q). Using Sobolev estimates, we can use this to control

$$\|e^{-it\Delta}f\|_{L^a_tL^c_x} \lesssim \||\nabla|^se^{-it\Delta}f\|_{L^a_tL^b_x} \lesssim \|f\|_{\dot{H}^s_x}, \qquad \frac{2}{a}+\frac{3}{c}=\frac{3}{2}-s.$$

3. Modified scattering

This is largely inspired from [7]. Here we consider the cubic Hartree equation in $\mathbb{R}^3$

$$(i\partial_t-\Delta)\,u+((-\Delta)^{-1}|u|^2)u=0, \qquad u(t=0)=\phi \tag{3.1}$$

and we show that, for small and localized initial data ϕ, the solution satisfies a different asymptotic behavior: modified scattering.

We can start with the same considerations as for (2.1). The main conservation laws are similar

$$\begin{aligned}M(u)=M(u_0), \qquad E(u)&=\int|\nabla_xu|^2+\frac{1}{2}\iint\frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy\\ &=\|\nabla_xu\|^2_{L^2}+\frac{1}{2}\||\nabla|^{-1}(|u|^2)\|^2_{L^2}\end{aligned} \tag{3.2}$$

and the scaling invariance

$$u(x,t)\to\lambda^2u(\lambda x,\lambda^2t)$$

which leaves invariant the $\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$ norm of the initial data. We see that both conservation laws are subcritical and we do not expect them to be very helpful in the analysis of the asymptotic behavior[7]. The scaling analysis would suggest to use a norm like

$$\|u\|_{Z'} := \|u\|_{L^2_t L^3_x}$$

as our main control norm. Unfortunately, this norm is not even finite for linear solutions (see Lemma 3.1). We will replace it by the weak control norm

$$\|u\|_Z := \|\widehat{u}\|_{L^\infty_t L^3_\xi} \tag{3.3}$$

which still controls the main term in the decay estimate.

Remark 3.1. *We can observe that the proof of weak linear scattering no longer works since this time, the best that can be said using the conservation laws is that*[8]

$$\|((-\Delta)^{-1}|u|^2)u\|_{L^{\frac{6}{5}}} \lesssim \|u\|_{L^2} \cdot \|(-\Delta)^{-1}|u|^2\|_{L^3} \lesssim \|u\|_{L^2} \cdot \||u|^2\|_{L^1}$$

which is insufficient since for a Schwartz function $\|e^{it\Delta}\phi\|_{L^6} \lesssim \langle t\rangle^{-1}$ *is not integrable. We will see indeed that the solutions follow a different asymptotic behavior:* modified *scattering.*

It turns out that there is a replacement for a conservation law coming from the invariance under Galilean translations

$$u(x,t) \to \mathcal{G}_v u(x,t) := e^{i\left[t|v|^2 - \langle v,x\rangle\right]} u(x-2tv,t) \tag{3.4}$$

which generates the following Galilean vector field

$$\begin{aligned} G_j u(x,t) &= \left(\frac{\partial}{\partial v^j}\mathcal{G}_v u\right)_{|v=0} = -2t\partial_{x^j} u - ix_j u(x,t) \\ &= -2te^{-i\frac{|x|^2}{4t}}\partial_{x^j}\left\{e^{i\frac{|x|^2}{4t}}u\right\} \end{aligned} \tag{3.5}$$

and one can see that

$$\|u_0\|_{\dot{H}^{-1}} \lesssim \|xu_0\|_{L^2} = \|G_j u(t=0)\|_{L^2}.$$

In addition, since (3.4) is a conservation law for the equation, we see that

$$0 = \left(\frac{\partial}{\partial v^j} eq(\mathcal{G}_v u)\right)_{|v=0} = \mathcal{L}_u G_j u$$

where $\mathcal{L}_u$ is the linearization

$$\mathcal{L}_u w := (i\partial_t - \Delta)\, w + \left[(-\Delta)^{-1}(|u|^2)\right] \cdot w + 2\left[(-\Delta)^{-1}\Re\left\{\overline{u}w\right\}\right] \cdot u \tag{3.6}$$

and once again, if we control a critical norm, we can hope that $\mathcal{L}_u$ has nice boundedness properties, which allows to propagate bounds on $G_j u$, which can then be interpolated with the conservation laws to obtain boundedness of the critical norm. This is amenable to a bootstrap analysis.

We first observe that the linearized equation (3.6) almost preserves the L^2 norm:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2} &= \Im\langle i\partial_t w, w\rangle = \Im\langle \mathcal{L}_u w, w\rangle - 2\Im\langle(-\Delta)^{-1}\Re\{\overline{u}w\}u, w\rangle \\ &= \frac{1}{2\pi}\iint_{\mathbb{R}^3} \frac{\Re\{\overline{u}(x)w(x)\}\Im\{\overline{u}(y)w(y)\}}{|x-y|} dxdy \end{aligned}$$

[7] Though of course, they can be used to guarantee global existence.

[8] And even this requires a missing endpoint Sobolev estimate.

and using Cauchy-Schwarz and Sobolev's embedding, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 &\lesssim \|w(x)w(y)\|_{L^2_{x,y}}\cdot\left(\iint\frac{|u(x)|^2|u(y)|^2}{|x-y|^2}dxdy\right)^{\frac{1}{2}}\\ &\lesssim \|w\|_{L^2_x}^2\||\nabla|^{-\frac{1}{2}}|u|^2\|_{L^2_x}\lesssim\|w\|_{L^2}^2\||u|^2\|_{L^{\frac{3}{2}}_x}\\ &\lesssim \|w\|_{L^2_x}^2\|u\|_{L^3_x}^2\end{aligned}$$

and Gronwall inequality gives that

$$\|w(t)\|_{L^2_x}^2\le\|w(0)\|_{L^2_x}^2\cdot\exp(C\|u\|^2_{L^2_t([0,t]:L^3_x)}). \tag{3.7}$$

This is almost enough to conclude. Looking at (3.5) suggests that a better unknown than u could be $v=e^{i|x|^2/4t}u$, for which we can observe, using Sobolev embedding that

$$\|u(t)\|_{L^3}=\|v\|_{L^3}\lesssim\|v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}}\le\|u\|_{L^2}^{\frac{1}{2}}\cdot(\frac{1}{t}\|Gu\|_{L^2})^{\frac{1}{2}}\lesssim t^{-\frac{1}{2}}M(u)^{\frac{1}{4}}\|Gu\|_{L^2}^{\frac{1}{2}}.$$

From which we deduce that

$$\|u\|_{L^2_t([1,t]:L^3_x)}\lesssim M(u)^{\frac{1}{4}}\|Gu\|_{L^\infty_t([0,t]:L^2_x)}\sqrt{\ln(t)}$$

which barely misses to close when combined with (3.7): assuming that $\|Gu\|_{L^2}$ remains bounded, one obtains that it increases slowly (if the data are small).

Thus a naïve bootstrap fails. Ultimately, the above scheme would not allow us to control a scale invariant norm uniformly in time, which makes it improbable. We will improve the above strategy by introducing a (weak) scale-invariant quantity, the Z-norm in (3.3) which we will show remains bounded along the evolution. As a reality check on it, we observe that it does remain bounded for linear evolutions and that it is weaker than the norms from the preliminary bootstrap (see (3.9)).

3.0.1. *General strategy.* Since we are considering a small data, setting $u=\varepsilon v$, we can rewrite the equation as

$$(i\partial_t-\Delta)\,v=\varepsilon^2\left[(-\Delta)^{-1}|v|^2\right]\cdot v$$

and, as a first approximation, one can integrate exactly the blue terms by setting $f(t)=e^{it\Delta}v(t)$. The equation then becomes

$$i\partial_t f(t)=-i\varepsilon^2e^{it\Delta}\left\{\left[(-\Delta)^{-1}|e^{-it\Delta}f|^2\right]e^{-it\Delta}f\right\}=\mathbf{T}(f,f,f).$$

which is a priori $O(\varepsilon^2)$ and so f remains small for times $O(\varepsilon^{-2})$, but the right-hand side is not purely perturbative and we need to extract an effective dynamics and recast

$$i\partial_t\widehat{f}-\frac{1}{t}\mathcal{T}_{eff}(\widehat{f},\widehat{f},\widehat{f})=\langle t\rangle^{-1-\delta}\mathcal{T}_{pert}(\widehat{f},\widehat{f},\widehat{f}) \tag{3.8}$$

and we expect the right-hand side terms to be perturbative, so that solutions asymptotically only solve the blue dynamics: they are well approximated by solutions of the form

$$i\partial_s g=\mathcal{T}_{eff}(g,g,g)$$

The Z-norm is then chosen as a conservation law for this dynamics, and a good model for the solutions to (3.1) is

$$u(t)=e^{-it\Delta}f(t)=e^{-it\Delta}\left(\check{g}(\ln(t))+o(1)\right).$$

If the dynamics for g converges, we obtain a reformulation of the scattering statement. Else, we obtain *modified* scattering since the main evolution (in time t) is the linear flow, but it needs to be composed with a secondary slower evolution (in time $\ln(t)$).

In our case, we can even integrate exactly the dynamics on the right-hand side of (3.8) by a simple phase conjugation by setting $g(t) = e^{i\Theta}\widehat{f}(t)$, and the final model can be given in terms of a modifed phase

$$u(x,t) = \int_{\mathbb{R}^3} g_\infty(\xi)e^{-i[\langle x,\xi\rangle+\Psi(\xi,t)]}d\xi, \qquad \dot{\Psi}(\xi,t) = |\xi|^2 + \frac{1}{t}\left[(-\Delta)^{-1}|g_\infty|^2\right](\xi).$$

We also refer to [4, 6] for adaptations of this method in other contexts, and to [5] for a different take on a similar problem.

3.1. **Small data modified scattering for Hartree.**

Theorem 3.1. *Assume that*

$$\|xu_0\|_{L^2_x} + \|u_0\|_{H^1_x} \le \varepsilon_0$$

then, there exists a unique global solution of (3.1) *which satisfies the bounds*

$$\|u\|_{L^\infty_t H^1_x} + \|u\|_Z \lesssim \varepsilon_0, \qquad \|Gu(t)\|_{L^2_x} \lesssim \varepsilon_0\langle t\rangle^{\varepsilon_0}$$

and this solution satisfies modified scattering in the sense that there exists a function $\widehat{g}_\infty \in L^2_x \cap L^3_x$ *such that*

$$\|\widehat{u}(\xi,t) - e^{i\left[t|\xi|^2+\ln(t)\left[(-\Delta)^{-1}|g_\infty(\xi)|^2\right]\right]}g_\infty(\xi)\|_{L^3_\xi} \to 0, \qquad \text{as } t\to+\infty.$$

3.2. **Closing the bootstrap.**

We need to find a modification of the above argument. Allowing growth of $\|u\|_{L^2_t([1,t]:L^3_x)}$ faster than logarithmic would not do: this would lead to exponential bounds after one round of the bootstrap and could not be saved. Instead, we will allow $\|Gu(t)\|_{L^2_x}$ to grow slowly and try to keep the $L^2_tL^3_x$ from growing faster than logarithmically.

Let

$$\|u\|_{X_t} := \|u\|_{L^2} + \|Gu\|_{L^2}.$$

Note that the X_t norm depends on t through the definition of G. Local well-posedness ensures that solutions belong to $C^0_tX_t$, and that $t\mapsto\|u(t)\|_{X_t}$ is continuous in time.

We can first observe that the weak norm Z is indeed weaker (than X). We start from the following reinterpretation of G which follows from (3.5):

$$\mathcal{F}\left\{G_je^{-it\Delta}f\right\}(\xi) = e^{it|\xi|^2}\partial_{\xi_j}\widehat{f}(\xi), \qquad \|Ge^{-it\Delta}f\|_{L^2} = \|xf\|_{L^2}.$$

and we conclude that

$$\|\widehat{u}\|_{L^3} = \|\widehat{f}\|_{L^3} \lesssim \|f\|_{L^{\frac{3}{2}}} \lesssim \|f\|_{L^2}^{\frac{1}{2}}\|xf\|_{L^2}^{\frac{1}{2}}. \tag{3.9}$$

To obtain quantitative bounds, it will suffice to prove the following bootstrap: Assume that, for $0\le t\le T$, there holds that

$$\begin{aligned}\|u_0\|_{L^2} + \|xu_0\|_{L^2} &\le \varepsilon_0,\\ \|u(t)\|_{L^2} + \|\widehat{u}(t)\|_{L^3} &\le \varepsilon_1,\\ \|Gu(t)\|_{L^2} &\le \varepsilon_1\langle t\rangle^\delta\end{aligned} \tag{3.10}$$

then, in fact

$$\begin{aligned}\|u(t)\|_{L^2} &\lesssim \varepsilon_0,\\ \|Gu(t)\|_{L^2} &\lesssim \varepsilon_0\langle t\rangle^{C\varepsilon_1^2},\\ \|\widehat{u}(t)\|_{L^3} &\lesssim \varepsilon_0+\varepsilon_1^3.\end{aligned} \tag{3.11}$$

Taking $\varepsilon_1 \gg \varepsilon_0$, this allows to propagate the bounds (3.11) by continuity globally in time.

The first bound in (3.11) follows by conservation of the energy. Using Lemma 3.1 and integrating in time, we find that

$$\|u\|_{L^2_t([1,T]:L^3_x)} \lesssim \sqrt{\ln(T)}\|\widehat{u}\|_{L^\infty_t L^3_\xi} + \sup_{0\le t\le T} t^{-\delta}\|u\|_X \lesssim \varepsilon_1\left[1+\sqrt{\ln(T)}\right].$$

which, given (3.7) provides the second bound in (3.11). Finally, the last bound is proven in Section 3.3.

3.2.1. *Precised dispersion inequality.* We prove the following precised dispersion inequality

Lemma 3.1. *Assume that f is a Schwarz function, then*

$$\|e^{-it\Delta}f\|_{L^3_x(\mathbb{R}^3)} = ct^{-\frac{1}{2}}\|\widehat{f}\|_{L^3(\mathbb{R}^3)} + t^{-\frac{3}{4}}\rho(t), \qquad |\rho(t)|\lesssim \|f\|_X, \tag{3.12}$$

and in particular, for $t\ge 2$, $\|e^{-it\Delta}f\|_{L^2([1,t]:L^3_x)} \gtrsim \sqrt{\ln(t)}$.

Proof of Lemma 3.1. We revisit the proof of the dispersion inequality

$$\left(e^{-it\Delta}f\right)(x) = \frac{1}{(4\pi it)^{\frac{3}{2}}}\int_{\mathbb{R}^3} e^{-i\frac{|x-y|^2}{4t}}f(y)dy = \frac{e^{-i\frac{|x|^2}{4t}}}{(4\pi it)^{\frac{3}{2}}}\int_{\mathbb{R}^3} e^{i\langle x,\frac{y}{4t}\rangle - i\frac{|y|^2}{4t}}f(y)dy.$$

If f is compactly supported, we see that $e^{-i\frac{|y|^2}{4t}}f(y)\to f(y)$; extracting the limit, we obtain

$$\begin{aligned}\left(e^{-it\Delta}f\right)(x) &= \frac{e^{-i\frac{|x|^2}{4t}}}{(2it)^{\frac{3}{2}}}\left(\widehat{f}(-\frac{x}{4t}) + \int_{\mathbb{R}^3} e^{i\langle\frac{x}{4t},y\rangle}f(y)\cdot\left(e^{-i\frac{|y|^2}{4t}}-1\right)dy\right)\\ &= \frac{e^{-i\frac{|x|^2}{4t}}}{(2it)^{\frac{3}{2}}}\left(\widehat{f}(-\frac{x}{4t})+R(x,t)\right)\end{aligned}$$

We will see that R is a remainder. We can start by inspecting the main term

$$\|\frac{e^{-i\frac{|x|^2}{4t}}}{(2it)^{\frac{3}{2}}}\widehat{f}(-\frac{x}{4t})\|_{L^3_x} = ct^{-\frac{3}{2}}\|\widehat{f}(-x/4t)\|_{L^3_x} = c't^{-\frac{1}{2}}\|\widehat{f}\|_{L^3_x}. \tag{3.13}$$

For the remainder[9], we note that

$$R[f](x,t) = \mathcal{F}\left\{(e^{-i\frac{|y|^2}{4t}}-1)f(y)\right\}(\frac{x}{4t})$$

and, decomposing, for $A:=\langle t\rangle^{\frac{1}{2}}$,

$$f = f_s+f_n, \qquad f_s(y):=\varphi(A^{-1}y)f(y), \qquad f_n(y):=(1-\varphi(A^{-1}y))f(y)$$

[9]We are thankful to A. Stingo for suggestimg this approach.

we see that

$$\|R[f_s]\|_{L^3} \lesssim t\|\mathcal{F}\left\{(e^{-i\frac{|y|^2}{4t}}-1)f_s\right\}\|_{L^3} \lesssim t\|(e^{-i\frac{|y|^2}{4t}}-1)f_s\|_{L^{\frac{3}{2}}} \lesssim A\|xf_s\|_{L^{\frac{3}{2}}} \lesssim A^{\frac{3}{2}}\|xf\|_{L^2},$$

$$\|R[f_n]\|_{L^3} \lesssim t\|(e^{-i\frac{|y|^2}{4t}}-1)f_n\|_{L^{\frac{3}{2}}} \lesssim t\|f_n\|_{L^{\frac{3}{2}}} \lesssim t\|xf\|_{L^2}\||y|^{-1}\mathbf{1}_{\{|y|\geq A\}}\|_{L^6} \lesssim tA^{-\frac{1}{2}}\|xf\|_{L^2}.$$

□

3.3. **Improved bootstrap.** In order to close the bootstrap (3.11), it suffices to control the third norm. Recall that the linear profile

$$f(t) := e^{it\Delta}u(t)$$

satisfies

$$i\partial_t f(t) = -e^{it\Delta}\left\{\left[(-\Delta)^{-1}|e^{-it\Delta}f|^2\right]e^{-it\Delta}f\right\}.$$

This is slightly easier to interpret in the Fourier space

$$\begin{aligned} i\partial_t \widehat{f}(\xi,t) &= -\iint_{\mathbb{R}^3} e^{-it\Phi(\xi,\eta,\sigma)}\widehat{f}(\xi-\eta,t)\overline{\widehat{f}}(\xi-\eta-\sigma,t)\widehat{f}(\xi-\sigma,t)\frac{d\eta d\sigma}{|\eta|^2},\\ \Phi(\xi,\eta,\sigma) &= |\xi|^2+|\xi-\eta-\sigma|^2-|\xi-\eta|^2-|\xi-\sigma|^2 = 2\langle\eta,\sigma\rangle \end{aligned}$$

which finally gives the evolution equation for the linear profile

$$\begin{aligned} i\partial_t \widehat{f}(\xi,t) &= -\iint_{\mathbb{R}^3} e^{-2it\langle\eta,\sigma\rangle}\widehat{f}(\xi-\eta,t)\overline{\widehat{f}}(\xi-\eta-\sigma,t)\widehat{f}(\xi-\sigma,t)\frac{d\eta d\sigma}{|\eta|^2},\\ &= \mathcal{T}(\widehat{f},\widehat{f},\widehat{f}). \end{aligned} \tag{3.14}$$

We can rewrite

$$\begin{aligned} \mathcal{T}(\widehat{f},\widehat{f},\widehat{f}) &= \iiint f(x)f(y)\overline{f}(z)e^{i\langle\xi,z-x-y\rangle}\cdot I\cdot dxdydz,\\ I &:= \iint e^{-2it\langle\eta,\sigma\rangle}e^{i\langle\eta,x-z\rangle}e^{i\langle\sigma,y-z\rangle}\frac{d\eta d\sigma}{|\eta|^2} \end{aligned}$$

and using the Fourier transform computations

$$\int_{\mathbb{R}^d} e^{-i\langle\xi,x\rangle}\frac{dx}{|x|^\alpha} = c_\alpha|\xi|^{\alpha-d}, \qquad 0<\alpha<d.$$

we can compute that

$$\begin{aligned} I &:= \int e^{i\langle\sigma,y-z\rangle}\left(\int e^{-i\langle\eta,z-x-2t\sigma\rangle}\frac{d\eta}{|\eta|^2}\right)d\sigma = c_2\int e^{i\langle\sigma,y-z\rangle}\frac{d\sigma}{|z-x-2t\sigma|}\\ &= \frac{c_2}{2t}\int e^{i\langle\sigma,y-z\rangle}\frac{d\sigma}{|\sigma-\frac{z-x}{2t}|} = \frac{c_2c_1}{2t}\frac{e^{-i\frac{1}{2t}\langle z-x,z-y\rangle}}{|z-y|^2} \end{aligned}$$

Now, on the support of f we expect $|z-x| \ll \sqrt{t}$, $|z-y| \ll \sqrt{t}$ (e.g. if we can assume that f is "close to" having compact support) and we can extract the main order contribution by suppressing the slowly oscillating exponential:

$$\mathcal{T}(\widehat{f},\widehat{f},\widehat{f}) = \frac{1}{t}\left[\mathcal{T}_{eff}(\widehat{f},\widehat{f},\widehat{f})+\mathcal{T}_{rem}(\widehat{f},\widehat{f},\widehat{f})\right]$$

where

$$\begin{aligned}\mathcal{T}_{eff}(\widehat{f},\widehat{f},\widehat{f}) &= \iiint f(x)f(y)\overline{f}(z)e^{i\langle\xi,z-x-y\rangle}\frac{dxdydz}{|z-y|^2} \\ &= \widehat{f}(\xi)\cdot\left(\iint f(y)\overline{f}(z)e^{i\langle\xi,z-y\rangle}\frac{dydz}{|z-y|^2}\right) \\ &= \widehat{f}(\xi)\cdot\left(\iint f(z-x)\overline{f}(z)e^{i\langle\xi,x\rangle}\frac{dxdz}{|x|^2}\right) \\ &= \widehat{f}(\xi)\cdot\left(\iint \widehat{f}(\eta)\overline{\widehat{f}}(\eta)e^{i\langle\xi-\eta,x\rangle}\frac{dxd\eta}{|x|^2}\right) \\ &= c_2\widehat{f}(\xi)\cdot\int|\widehat{f}(\eta)|^2\frac{d\eta}{|\xi-\eta|}\end{aligned}$$

while the remainder is give by

$$\begin{aligned}\mathcal{T}_{rem}(\widehat{f},\widehat{f},\widehat{f}) &= \iiint f(x)f(y)\overline{f}(z)e^{i\langle\xi,z-x-y\rangle}\frac{e^{-i\frac{1}{2t}\langle z-x,z-y\rangle}-1}{|z-y|^2}dxdydz \\ &= \iiint f(\alpha+z-y)f(y)\overline{f}(z)e^{-i\langle\xi,\alpha\rangle}\frac{e^{i\frac{1}{2t}\langle y-\alpha,y-z\rangle}-1}{|z-y|^2}d\alpha dydz\end{aligned}$$

and we can rewrite (3.14), isolating the perturbative terms:

$$i\partial_t\widehat{f}(\xi,t)-\frac{c}{t}\left[(-\Delta)^{-1}|\widehat{f}|^2\right](\xi)\cdot\widehat{f}(\xi,t)=\frac{1}{t}\mathcal{T}_{rem}(\widehat{f},\widehat{f},\widehat{f}).$$

the left-hand side can be integrated exactly by setting

$$\widehat{g}(\xi,t)=\widehat{f}(\xi,t)e^{i\Theta(\xi,t)},\qquad \dot{\Theta}(\xi,t)=\frac{1}{t}\left[(-\Delta)^{-1}|\widehat{g}|^2\right](\xi),$$

which satisfies

$$i\partial_t\widehat{g}(\xi,t)=\frac{1}{t}e^{i\Theta(\xi,t)}\mathcal{T}_{rem}(\widehat{f},\widehat{f},\widehat{f}),\tag{3.15}$$

with a right hand side which is perturbative according to Lemma 3.2.

Lemma 3.2. *There holds that*

$$\|\mathcal{T}_{rem}(\widehat{f_1},\widehat{f_2},\widehat{f_3})\|_{L^3}\lesssim\langle t\rangle^{-\delta}\prod_{j=1}^{3}\left[\|f_j\|_{L^2}+\|xf_j\|_{L^2}\right]$$

Proof of Lemma 3.2. Using Hausdorff-Young, and Hardy-Littlewood's inequalities, we see that

$$\begin{aligned}\|\mathcal{T}_{rem}(\widehat{f_1},\widehat{f_2},\widehat{f_3})\|_{L^3_\xi} &\lesssim \|\iint f_1(\alpha+z-y)f_3(y)\overline{f_2}(z)\frac{e^{i\frac{1}{2t}\langle y-\alpha,y-z\rangle}-1}{|z-y|^2}dydz\|_{L^{\frac{3}{2}}_\alpha} \\ &\lesssim \|f_1\|_{L^{\frac{3}{2}}}\cdot\left|\iint\frac{|f_3(y)\overline{f_2}(z)|}{|z-y|^2}dydz\right| \\ &\lesssim \|f_1\|_{L^{\frac{3}{2}}}\|f_2\|_{L^{\frac{3}{2}}}\|f_3\|_{L^{\frac{3}{2}}}\end{aligned}$$

We can decompose, for $A:=\langle t\rangle^{\delta}$,

$$f_i:=f_i^s+f_i^n,\qquad f_i^s(x)=\varphi(A^{-1}x)f_i(x),\qquad f_i^n(x)=(1-\varphi(A^{-1}x))f_i(x),$$

with

$$\|f_i^n\|_{L^{\frac{3}{2}}}\lesssim\|f_i^n\|_{L^2}^{\frac{1}{2}}\|f_i^n\|_{L^{\frac{6}{5}}}^{\frac{1}{2}}\lesssim A^{-\frac{1}{2}}\|xf\|_{L^2}$$

and we deduce that we may assume

$$f_i(x) = \varphi(A^{-1}x)f_i(x), \qquad A := \langle t\rangle^\delta, \qquad \|f_i\|_{L^1} \lesssim A^{\frac{3}{2}}\|f_i\|_{L^2} \lesssim A^{\frac{3}{2}}\sqrt{M(u)}$$

and when all positions are small, we can expand

$$\left|e^{i\frac{1}{2t}\langle y-\alpha, y-z\rangle} - 1\right| \lesssim t^{-1}A^2$$

which leads to an acceptable contribution. □

Finally, we can close the bootstrap estimate (3.11). Indeed, we can start from (3.15) and obtain that

$$\|\widehat{u}(t)\|_{L^3} = \|\widehat{f}\|_{L^3} = \|\widehat{g}\|_{L^3} \leq \|\widehat{g}(0)\|_{L^3} + \int_{s=0}^t \langle s\rangle^{-1-\delta}\varepsilon_1^3 ds \leq \varepsilon_0 + C\varepsilon_1^3.$$

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