EQUATIONS OF UNIVERSAL TORSORS AND COX RINGS

B. Hassett
Department of Mathematics, Rice University, MS 136, Houston, TX 77251-1892, U.S.A. • E-mail: hassett@math.rice.edu

Abstract. We discuss several constructions of universal torsors over rational surfaces.

1. Universal torsors and Cox rings

1.1. Motivating Example. All fields are supposed to be of characteristic 0.
Let $X/K$ be a quintic Del Pezzo surface over a number field $K$. We have $\overline{X} = X_{\overline{K}} = B\ell_{p_1, p_2, p_3, p_4} \mathbb{P}^2$, i.e. geometrically, $X$ is the blow-up of $\mathbb{P}^2$ in four points in general position. Without loss of generality, we may assume that

$$P_1 = [1, 0, 0], P_2 = [0, 1, 0], P_3 = [0, 0, 1], P_4 = [1, 1, 1].$$

Theorem 1 (Enriques, Swinnerton-Dyer). Even in the non-split case, $X(K) \neq \emptyset$.

Proof. See [Sko93].

Notes by Ulrich Derenthal, June 18th and 19th, 2004.
Since there is a unique projectivity taking arbitrary generic points $P_1, P_2, P_3, P_4$ (i.e., distinct and no three of them collinear) to $[1, 0, 0], \ldots, [1, 1, 1]$ as above, the geometry behind this over $\overline{X}$ is (where $P_5 \in \overline{X}$ is the point we want to describe):
\[
\overline{X} = \text{SL}_3 \backslash \{(P_1, \ldots, P_5) \in \mathbb{P}^2\} = \text{SL}_3 \backslash \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{P}^5 : \text{GL}_3 \backslash M(3 \times 5) \cong \text{Gr}(3, 5) \right\}.
\]

Consider the Grassmannian of $3$-dimensional subspaces of $5$-dimensional space $\text{Gr}(3, 5)$. Since such a subspace is described by a basis which is unique only up to an action of $\text{GL}_3$, we have $\text{GL}_3 \backslash M(3 \times 5) \cong \text{Gr}(3, 5)$, where we interpret the three rows of a $3 \times 5$ matrix as a basis. This implies that $\text{SL}_3 \backslash M(3 \times 5)$ is the cone over this Grassmannian.

Therefore, $\overline{X} \cong \text{Cone}(\text{Gr}(3, 5)) \bmod \mathbb{G}_m^5$. Here, $\text{Gr}(3, 5)$ is embedded into $\mathbb{P}^9$ by the Plücker embedding.

The “miracle” is that this generalizes to non-closed fields.

**Remark 2.** The permutation group $S_5$ of the five points acts on the situation, and actually $\text{Aut}(\overline{X}) = S_5$.

Descent data for $X$ is given by representations $\rho : \text{Gal}(\overline{K}/K) \to S_5$. Let $T_\rho$ be the nonsplit form of $\mathbb{G}_m^5$ corresponding to $\rho$. In fact, $X$ is $\text{Cone}(\text{Gr}(3, 5)) \bmod T_\rho$.

**1.2. Universal torsors.** Let $X$ be a smooth projective variety over $\overline{K}$. Assume $\text{Pic}(X)$ is free of rank $r$. Let $T_X$ be the Néron-Severi torus, i.e., its character group is $\chi^*(T_X) = \text{Pic}(X)$.

**Definition 3.** A universal torsor $\mathcal{U}$ is a $T_X$-principal homogeneous space

\[
\begin{array}{ccc}
T_X & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow \\
X & & 
\end{array}
\]

so that given an element $\lambda \in \chi^*(T_X)$ (i.e., $\lambda : T_X \to \mathbb{G}_m$), then $\mathcal{M}_\lambda \cong R_{\lambda} - \{0\}$ as $\mathbb{G}_m$-bundles over $X$. Here, $\mathcal{L}_\lambda \in \text{Pic}(X)$ is the line bundle associated to $\lambda$ by $\chi^*(T_X) \cong \text{Pic}(X)$, and $\mathcal{M}_\lambda$ is the associated bundle to the principal bundle $\mathcal{U} \to X$ induced by the representation $\lambda$.

**Example 4.** 1. Let $X = \mathbb{P}^n$. Then $\mathcal{U} = \mathbb{A}^{n+1} - \{0\}$ is the corresponding universal torsor with the torus acting diagonally. We have $\mathcal{U}/\mathbb{G}_m \cong X = \mathbb{P}^n$. 
2. Let $X$ be the quintic Del Pezzo surface as above with the action of $T_X$ on $\text{Cone(Gr(3,5))}$. Then the universal torsor $\mathcal{U}$ is the open subset of $\text{Cone(Gr(3,5))}$ on which $T_X$ acts freely.

The abstract approach to universal torsors is as follows: Choose a minimal set $\mathcal{L}_1, \ldots, \mathcal{L}_r$ generating $\text{Pic}(X)$ over $\mathbb{Z}$. Denote $\mathcal{L}_j - \{0\text{-section}\}$ by $\mathcal{L}_j^\times$. Let $\mathcal{U} = \mathcal{L}_1^\times \times \cdots \times \mathcal{L}_r^\times$. Then $T_X \rightarrow \mathcal{U} \rightarrow X$ is a $T_X$-principal bundle defining the universal torsor.

However, this abstract definition is not very useful, e.g., for number theoretic applications.

**Remark 5.** Over non-closed fields, we may not be able to descend the universal torsor $\mathcal{U}$.

For example, consider a non-split conic $X$. It is geometrically isomorphic to $\mathbb{P}^1$, but it has no line bundle isomorphic to $O_{\mathbb{P}^1}(1)$ over the ground field. It only has line bundles of even degree, so there cannot exist a universal torsor over the ground field.

### 1.3. Total coordinate rings / Cox rings.

**Definition 6.** Let $X$ be a projective variety with properties as above. Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be a basis of $\text{Pic}(X)$. Then the Cox ring of $X$ is defined as

$$\text{Cox}(X) = \bigoplus_{(n_1, \ldots, n_r) \in \mathbb{Z}^r} \Gamma(X, \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r}).$$

Properties of $\text{Cox}(X)$ are:

1. It is graded by $\text{Pic}(X)$: for $\lambda \in \chi^*(T_X) \cong \text{Pic}(X)$, the part of degree $\lambda$ is given by $\text{Cox}(X)_\lambda = \Gamma(X, \mathcal{L}_j)$.
2. The torus $T_X$ acts naturally on $\text{Cox}(X)$: For $t \in T_X$, $s \in \text{Cox}(X)_\lambda$, this action is given by $t(s) := \lambda(t) \cdot s$.
3. $\text{Cox}(X)$ is independent of the choice of generators $\mathcal{L}_i$ of the Picard group. Given two sets of generators $\mathcal{L}_i$ and $\mathcal{M}_j$, the induced isomorphism of rings is canonical only up to the action of the torus $T_X$. The reason is that the isomorphism depends on a choice of isomorphisms $L_j \cong M_1^{\otimes n_1} \otimes \cdots \otimes M_r^{\otimes n_r}$, $j \in \{1, \ldots, r\}$.

However, such an isomorphism is not canonical: $\mathcal{L}_j$ has automorphisms given by scalar multiplication. For details, see [HT04].

The existence of non-trivial automorphisms makes the descent of universal torsors an interesting question.
4. The graded pieces of \( \text{Cox}(X) \) which are non-zero correspond to effective divisors on \( X \).

The Cox ring does not need to be finitely generated:

**Example 7 (Mukai).** Let \( X = \text{Bl}_{p_1, \ldots, p_n} \mathbb{P}^{r-1} \) be the blowup of projective space in \( n \) points in general position. If \( \frac{1}{r} + \frac{1}{n} + \frac{1}{m} \geq 1 \), then \( \text{Cox}(X) \) is not finitely generated (i.e., for \( \mathbb{P}^2 \): \( n \geq 9 \); for \( \mathbb{P}^3 \): \( n \geq 8 \)). Details can be found in [Muk01].

However, it is finitely generated if one of the following conditions is true:
1. The cone of effective divisors \( \text{NE}(X) \) is generated by a finite collection of semi-ample line bundles (e.g., \( X = G/P \) where \( P \) is parabolic subgroup of an algebraic group \( G \)).
2. \( X \) is (log) Fano of dimension \( \leq 3 \).
3. \( X \) is toric. In this case, for \( X - \mathcal{O}_{m}^{\dim X} = \bigcup_{j=1}^{N} D_{j} \) where the \( D_{j} \) are subvarieties of codimension 1, and \( s_{j} \in \Gamma(\mathcal{O}_{X}(D_{j})) \) is non-zero, then \( \text{Cox}(X) \cong K[s_{1}, \ldots, s_{N}] \).

1.4. Relations between universal torsors and Cox rings. From now on, assume that \( \text{Cox}(X) \) is finitely generated. Let \( \mathcal{V} = \text{Spec}(\text{Cox}(X)) \). It is affine with \( T_X \)-action \( T_X \times \mathcal{V} \to \mathcal{V} \). Fix an open subset \( \mathcal{U} \) on which \( T_X \) acts freely. The basic fact is that \( \mathcal{U} \) is a \( T_X \)-principal bundle over \( X \):

\[
\begin{array}{ccc}
T_X & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow \\
X & \longleftarrow & \mathcal{U}
\end{array}
\]

and \( \mathcal{U} \) is a universal torsor.

The punchline is that this way, the universal torsor \( \mathcal{U} \) is naturally a quasi-affine variety. Therefore, giving equations for \( \mathcal{U} \) is equivalent to giving generators and relations for \( \text{Cox}(X) \). This can be done by algebro-geometric methods, which may be seen as an improvement to the existing number theoretic method to calculate universal torsors.

To sketch a proof of these results, observe that \( X \) is naturally a Geometric Invariant Theory quotient \( (\mathbb{V}///T_X)_{\lambda} \) (by Keel–Hui, [HK00]) after specifying a linearization \( \lambda \in \chi^{*}(T_X) \) so that \( \mathcal{Z}_{X} \) is an ample line bundle on \( X \).

Note that we need to mix affine invariant theory and the usual projective Geometric Invariant Theory to interpret \( (\mathbb{V}///T_X)_{\lambda} \): First take the affine quotient under the action of \( \ker(\lambda) \), which gives an affine variety. Then take \( \text{Proj} \) using the grading coming from the character \( \lambda \).
Then \( \text{Proj}(\bigoplus_{n \geq 0} \text{Cox}(X)_{n\lambda}) = (\mathcal{V}'/\mathcal{T}_X)_\lambda \) by Geometric Invariant Theory. The left hand side is \( \text{Proj}(\bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}_\lambda^\otimes n)) \), which is just \( X \) since \( \mathcal{L}_\lambda \) is ample.

A second observation is that given \( \lambda \in \chi^*(\mathcal{T}_X) \), i.e., \( \lambda : \mathcal{T}_X \to \mathbb{G}_m \), the associated bundle induces \( \mathcal{L}_\lambda^{-1} \). Therefore, it suffices to check the claim for ample \( \lambda \).

We have an inclusion \( \bigoplus_{n \geq 0} \text{Cox}(X)_{n\lambda} \to \text{Cox}(X) \) which induces a dominant map \( \mathcal{V}' \to \text{Cone}(X \subset P^N, \mathcal{L}_\lambda) \). Therefore, we have

\[
\begin{array}{c c c}
\mathcal{V}' & \longrightarrow & \text{Cone}(X \subset P^N) \\
\uparrow & & \uparrow \\
\mathcal{V} & \longrightarrow & (\text{Cone}(X \subset P^N) - \{0\}) \cong (\mathcal{L}_\lambda^{-1})^* \\
\end{array}
\]

The point is: One gets hold of the universal torsor by embedding it into the affine variety \( \text{Spec}(\text{Cox}(X)) \).

2. Equations of universal torsors

From now on, let \( X \) be a smooth projective variety over an algebraically closed field \( K \) of characteristic 0 with Pic\((X) \cong \mathbb{Z} \) whose Cox ring is finitely generated. Therefore, the cone of effective divisors \( \text{NE}(X) \) is finitely generated.

2.1. The method of Colliot-Thélène and Sansuc. This approach to the calculation of Cox rings can be found in [CTS87].

On \( X \), choose effective divisors \( D_1, \ldots, D_N \) generating Pic\((X) \). Let \( W = X \setminus (D_1 \cup \cdots \cup D_N) \). Since removing these generators kills the Picard group, Pic\((W) = 0 \).

We have an exact sequence

\[
0 \to K[W]^*/K^* \to \bigoplus_{j=1}^N \mathbb{Z}D_j \to \text{Pic}(X) \to 0
\]

where \( K[W]^*/K^* \) describes the linear equivalences among \( \{D_1, \ldots, D_N\} \).

Dualizing this sequence by applying \( \text{Hom}(\cdot, \mathbb{G}_m) \), we obtain

\[
1 \to \mathcal{T}_X \to \mathbb{G}_m^N \xrightarrow{\partial} R_W \to 1.
\]
Remark 8. A morphism $\varphi : Z \to R_W$ gives a $T_X$-torsor:

$$
\begin{align*}
T_X &\longrightarrow G^N_m \times_{R_W} Z \quad \supset \quad q^{-1}(\varphi(z)) \\
&\downarrow \\
Z &\quad \ni \quad z
\end{align*}
$$

The strategy is to construct a $T_X$-torsor $\mathcal{U}_W$ over $W$ which extend to a universal torsor over $X$. This strategy works well in many cases, but not in general.

The morphism $\varphi : W \to R_W$ is constructed by constructing a splitting $\sigma$ to the quotient

$$
K[W]^* \xrightarrow{\sigma} K[W]^*/K^* : 
$$

Note that $\sigma$ induces a $K$-algebra homomorphism

$$
K[R_W] = K[t_1, t_1^{-1}, \ldots, t_{N-r}, t_{N-r}^{-1}] \to K[W], \quad t_j \mapsto \sigma(t_j),
$$

where the $t_j$ form a basis for $\chi^*(R_W)$ and $r = \text{Rank}(\text{Pic}(X))$. Since $R_W$ is affine, such a homomorphism corresponds to a $K$-morphism $W \to R_W$, which defines $\varphi$.

The key fact is that the morphism $\varphi$ extracted from $\sigma$ gives a torsor $T_X \to \mathcal{U}_W \to W$ on $W$ admitting an extension to a universal torsor $T_X \to \mathcal{U} \to X$ over $X$.

$$
\begin{align*}
T_X &\longrightarrow \mathcal{U}_W \\
&\downarrow \\
W &\quad \ni \quad X
\end{align*}
$$

An explicit method for constructing such an extension is not known. Only the existence is proven in [CTS87].

Remark 9 (Batyrev). Given a point $P \in W$, we get a natural splitting $\sigma_P : K[W]^*/K^* \to K[W]^*$: for every element of $K[W]^*/K^*$, choose a representing $f$ satisfying $f(P) = 1$.

2.2. The example of the quintic Del Pezzo surface. Let $X = \text{Bl}_{P_1, \ldots, P_4} \mathbb{P}^2$ be again the blow-up of $\mathbb{P}^2$ in

$$
P_1 = [1, 0, 0], \quad P_2 = [0, 1, 0], \quad P_3 = [0, 0, 1], \quad P_4 = [1, 1, 1].
$$

We will see how to obtain the Plücker equations defining the universal torsor by this method.
Consider the exceptional divisors $E_i$ and the transforms $l_{ij}$ of the lines through $P_i$ and $P_j$ ($i \neq j \in \{1, \ldots, 4\}$). Choose coordinates $[x, y, z]$ and let $u = \frac{x}{z}, v = \frac{y}{z}$. Consider

\[
\text{div}(u = x/z) = l_{23} + E_3 - l_{12} - E_1, \\
\text{div}(v = y/z) = l_{13} + E_3 - l_{12} - E_2, \\
\text{div}(u - 1) = l_{24} + E_4 - l_{12} - E_1, \\
\text{div}(v - 1) = l_{14} + E_4 - l_{12} - E_2, \\
\text{div}(u - v) = l_{34} + E_3 + E_4 - l_{12} - E_1 - E_2.
\]

Next, we normalize these functions by constructing a section $\sigma_P$ from a chosen point, say $P = [3, 2, 1]$. This gives a morphism $\phi : W \to R_W$ as above.

Consider the sections $\lambda_{ij}$ corresponding to $l_{ij}$ and $\eta_i$ to $E_i$. Using the normalization, we obtain:

\[
\frac{u}{3} = \frac{\lambda_{23} \eta_3}{\lambda_{12} \eta_1}, \quad \frac{v}{2} = \frac{\lambda_{13} \eta_3}{\lambda_{12} \eta_2}, \quad \frac{u - 1}{2} = \frac{\lambda_{24} \eta_4}{\lambda_{12} \eta_1}, \quad \frac{v - 1}{2} = \frac{\lambda_{14} \eta_4}{\lambda_{12} \eta_2}, \quad \frac{u - v}{2} = \frac{\lambda_{34} \eta_3 \eta_4}{\lambda_{12} \eta_1 \eta_2}.
\]

Then the relations between the sections $u, v, u - 1, v - 1, u - v$ give relations between the sections $\lambda_{ij}, \eta_i$:

\[
3 \frac{u}{3} - 2 \frac{v}{2} = u - v \quad \leadsto \quad -(3\lambda_{23})\eta_2 + (2\lambda_{13})\eta_1 + \lambda_{34}\eta_4 = 0, \\
2 \frac{v}{2} = (v - 1) + 1 \quad \leadsto \quad \lambda_{14}\eta_4 - (2\lambda_{13})\eta_3 + \lambda_{12}\eta_2 = 0, \\
2 \frac{u - 1}{2} - (v - 1) = u - v \quad \leadsto \quad \lambda_{34}\eta_3 - (2\lambda_{24})\eta_4 + \lambda_{14}\eta_1 = 0, \\
3 \frac{u}{3} = 2 \frac{u - 1}{2} + 1 \quad \leadsto \quad (2\lambda_{24})\eta_4 - (3\lambda_{23})\eta_3 + \lambda_{12}\eta_1 = 0, \\
-(u - v) + v(u - 1) - (v - 1)u = 0 \quad \leadsto \quad \lambda_{12}\lambda_{34} - (2\lambda_{13})(2\lambda_{24}) + (3\lambda_{23})\lambda_{14} = 0.
\]

Replacing $3\lambda_{23}, 2\lambda_{13}, 2\lambda_{24}$ by new variables exactly gives the Plücker relations.

2.3. The Cox ring approach. Consider a different example:

\[X = Bl_{P_1, P_2, P_3} \mathbb{P}^2\] where $P_1 = [1, 0, 0], P_2 = [1, 1, 0], P_3 = [0, 1, 0], \]
i.e., $X$ is the blow-up of $\mathbb{P}^2$ in three points lying on a line. Let $l_{123}$ be the transform of this line.

Basic facts on $X$ are:

1. $\text{NE}(X) = \langle l_{123}, E_1, E_2, E_3 \rangle$ is a simplicial cone, i.e., there are no relations between its generators. Therefore, the previous method does not work.
We have $W = X - \{E_1, E_2, E_3, l_{123}\} \cong \mathbb{A}^2$, and $X$ is an equivariant compactification of $\mathbb{G}_m^2$ acting on $\mathbb{A}^2$ by translation.

2. The ample cone, which is the dual of the effective cone, is generated by

$$\{l_{123} + E_1 + E_2 + E_3, l_{123} + E_1 + E_2, l_{123} + E_1 + E_3, l_{123} + E_2 + E_3\}.$$

3. The anticanonical divisor $-K_X$ is nef and big. Therefore, $X$ is (log) Del Pezzo.

Next, we are looking for generators and relations of $\text{Cox}(X)$. Generators are $\lambda_{123} \in \Gamma(\mathcal{O}_X(l_{123})) \subseteq \text{Cox}(X)_{l_{123}}$, which is vanishing exactly along $l_{123}$, and $\eta_j \in \Gamma(\mathcal{O}_X(E_j)) \subseteq \text{Cox}(X)_{E_j}$ for $j \in \{1, 2, 3\}$.

These sections do not generate the Cox ring — in cases where they generate it, the method of Colliot-Thélène works well, but not here. We must choose additional generators: $\Gamma(\mathcal{O}_X(l_{123} + E_1 + E_2))$ corresponds to linear forms in $x, y, z$ vanishing at $P_3$, i.e., it is $K^2 \cong \langle x, z \rangle$. Besides $\lambda_{123}\eta_1\eta_2$, which can be identified as $z$, we can choose another section $\xi_3$ such that $\xi_3\eta_3 = -x$.

Similarly, we have $\xi_1 \in \Gamma(\mathcal{O}_X(l_{123} + E_2 + E_3))$ such that $y = \xi_1\eta_1$ and $\xi_2 \in \Gamma(\mathcal{O}_X(l_{123} + E_1 + E_3))$ such that $x - y = \xi_2\eta_2$.

This gives a homomorphism

$$\psi : K[\lambda_{123}, \eta_1, \eta_2, \eta_3, \xi_1, \xi_2, \xi_3]/\langle \eta_1\xi_1 + \eta_2\xi_2 + \eta_3\xi_3 \rangle \rightarrow \text{Cox}(X),$$

and since the dimension of both of these is $6$ ($\dim(X) = 2$ and $\text{Rank} (\text{Pic}(X)) = 4$), it is reasonable to hope that this is an isomorphism.

**Remark 10.** Then $\eta_1\xi_1 + \eta_2\xi_2 + \eta_3\xi_3$ is the equation of the universal torsor $T_X \rightarrow \mathcal{Y} \rightarrow X$ in the sense that

$$\mathcal{Y} \subseteq \mathcal{Y}' := \text{Spec} K[\lambda_{123}, \eta_1, \eta_2, \eta_3, \xi_1, \xi_2, \xi_3]/\langle \eta_1\xi_1 + \eta_2\xi_2 + \eta_3\xi_3 \rangle.$$

**Strategy of the proof.** First, consider $\psi$ in degrees $\nu$ corresponding to a nef line bundles on $X$. Such line bundles are semi-ample and in this case even globally generated. By induction on the effective monoid or by application of a vanishing theorem, we can prove that $\psi$ is surjective in these nef degrees.

In degrees $\nu$ corresponding to not necessarily nef divisors $\nu$, we reduce to the nef case the following way: Given $s \in \text{Cox}(X)_\nu = \Gamma(X, \mathcal{L}_\nu)$, there exists a nef line bundle $m$, a section $\mu \in \text{Cox}(X)_m$ and $a, b_1, b_2, b_3 \in \mathbb{Z}_{\geq 0}$ so that $s = \mu \lambda_{123}^{a_1} \eta_1^{b_1} \eta_2^{b_2} \eta_3^{b_3}$. This follows from the geometric fact that, given effective $D$ on $X$, we can write $D = M + F$ for a base point free divisor $M$ and a fixed divisor $F$ supported in $\{l_{123}, E_1, E_2, E_3\}$. \[\Box\]
References


