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HARMONIC MAPS FROM A SIMPLICIAL COMPLEX AND GEOMETRIC RIGIDITY

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Abstract

We study harmonic maps from an admissible flat simplicial complex to a non-positively curved Riemannian manifold. Our main regularity theorem is that these maps are $C^{1,\beta}$ at the interfaces of the top-dimensional simplices in addition to satisfying a balancing condition. If we assume that the domain is a 2-complex, then these maps are C^∞. As an application, we show that the regularity, the balancing condition and a Bochner formula lead to rigidity and vanishing theorems for harmonic maps. Furthermore, we give an explicit relationship between our techniques and those obtained via combinatorial methods.

1. Introduction

Harmonic maps are critical points of the energy functional. The energy of a map $\varphi : X \to N$ between two spaces X and N is defined to be the integral over the domain space of the energy density function which measures the total stretch of the map at each point of X. In the case when X and N are smooth Riemannian manifolds, the energy density function is the squared norm of the differential of the map.

One of the highlights of the harmonic map theory has been its successful applications to study representation of discrete groups. Suppose Γ is a fundamental group of a manifold X acting on a space N by $\rho : \Gamma \to \mathrm{Isom}(N)$. The idea is to associate the action with an equivariant harmonic map $\tilde{f} : \tilde{X} \to N$ where $\tilde{X}$ is the universal cover of X. Once the existence is established, one can use the curvature assumptions on the domain and the target spaces to make strong statements about $\tilde{f}$ and hence about the representation ρ.

To illustrate this, let X be a compact Riemannian manifold of non-negative Ricci curvature and with fundamental group Γ, and N be a complete Riemannian manifold of non-positive sectional curvature. Consider a representation $\rho : \Gamma \to \mathrm{Isom}(N)$ and let $\tilde{f} : \tilde{X} \to N$ be a

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$\Gamma-$equivariant harmonic map. Such a map $\tilde{f}$ exists as long as the action ρ does not fix an equivalence class of rays in N. The Eells-Sampson Bochner formula implies

$$\frac{1}{2}\triangle|\nabla\tilde{f}|^2 = |\nabla d\tilde{f}|^2 + \langle d\tilde{f}(\mathrm{Ric}^X(e_k)), d\tilde{f}(e_k)\rangle_{\tilde{f}^{-1}TN} - \langle R^N(d\tilde{f}(e_k), d\tilde{f}(e_l))d\tilde{f}(e_l), d\tilde{f}(e_k)\rangle_{\tilde{f}^{-1}TN} \tag{1}$$

where Ric^X and R^N are the Ricci and sectional curvatures of X and N respectively. The right hand side of the equation above is non-negative. Furthermore, Stoke's theorem says

$$\int_X \triangle|\nabla\tilde{f}|^2 = 0, \tag{2}$$

from which we conclude that each term on the right hand side of equation (1) is also zero. In particular, $\nabla d\tilde{f} = 0$, i.e., the map $\tilde{f}$ is totally geodesic. The representation $\rho : \Gamma \to \mathrm{Isom}(N)$ is then said to be rigid. Notice that the two important ingredients that were used here are the Bochner formula and Stoke's Theorem.

Further rigidity formulas were discovered by Siu, Corlette, and others; see for further examples [**Si**], [**C**], and [**MSiY**]. In the seminal work of Gromov-Schoen [**GS**] and subsequently Korevaar-Schoen [**KS1**], [**KS2**], the situation in which N is only a complete metric space rather than a smooth manifold was considered. This enabled them to prove superrigidity of p-adic representations along the lines of [**C**].

In a different direction, one can ask if it is possible to allow the domain X to be singular, for example a simplicial complex. This idea goes back to the work of Garland [**G**] and was subsequenctly elaborated by several groups of authors (cf. [**BS**], [**Gr**], [**IN**], [**W1**], [**W2**] and [**Z**]) in connection with Kazhdan property (T). For the nonlinear versions, the key idea is to define a combinatorial version of harmonic maps and relate them via a Bochner formula to a combinatorial analogue of curvature on X. This is in essence a refinement of Garland's notion of p-adic curvature. (For a definition of combinatorial harmonic maps, see [**J**] and [**W1**], [**W2**].)

The actual notion of a harmonic map on a polyhedral domain, rather than its combinatorial counterpart, was first introduced in [**Ch**] and was further developed in [**EF**], [**DM**] and [**M**]. In the special case when X is a flat n-dimensional simplicial complex, it was shown in [**DM**] and [**M**] that the harmonic map is Lipschitz across the edges. In the case of a 2-dimensional flat simplicial complex, the harmonic map has particular growth rate at the vertices depending on the order of the map. (See Section 2, Theorems 1 and 2 for precise statements of the results.)

The main goal of this paper is to further improve the regularity of harmonic maps defined on polyhedral domains. In particular, we show that the harmonic map must be $C^{1,\beta}$ across the strict $(n-1)$-skeleton and satisfy a natural balancing condition. (See Theorem 4 and Corollary 5). Furthermore, if $n = 2$, then we show that the harmonic map must be C^∞ at the interfaces of the 1-skeleton.

The second goal of the paper is to generalize the Bochner technique described earlier in the introduction to the case when X is a simplicial complex. Here, we restrict to the case $\dim X = 2$ and introduce weights (see the beginning of Section 2 for a precise definition.) This imposes no restriction as far as applications to group theory is concerned. We show that under the assumption that $|\nabla \tilde{f}|$ is bounded, Stoke's formula (2) still holds. By combining this with a simplex-wise Bochner formula (1), we obtain that $\tilde{f}$ is totally geodesic (see Theorem 8).

It therefore remains to establish the assumption on X for which harmonic maps from X are forced to satisfy the condition $|\nabla \tilde{f}| \leq C$. It is not hard to see that the latter condition is equivalent to the condition that $\mathrm{ord}_p \tilde{f} \geq 1$ for all points p in X, which in turn is equivalent to the condition that the first nonzero eigenvalue of the Laplacian of the link of p, $\mathrm{Lk}(p)$, is ≥ 1. Of course the $\mathrm{Lk}(p)$ is a graph and we can easily relate the spectrum of the Laplacian on $\mathrm{Lk}(p)$ to the spectrum of the discrete Laplacian (cf. Proposition 13 and Corollary 14). In particular, we show that the condition that the first nonzero eigenvalue being > 1 is equivalent to the first nonzero eigenvalue of the discrete Laplacian being $> \frac{1}{2}$, which is precisely the condition appearing in the combinatorial approach (cf. [**BS**], [**IN**], [**W1**], [**W2**], [**Z**]). This allows us to deduce the main theorem in [**W1**], that if Σ^n is compact simplicial n-complex with admissible weight whose first nonzero eigenvalue of each link of a vertex is $> \frac{1}{2}$ then any isometric action of Γ on a complete simply connected manifold of nonpositive sectional curvature has a fixed point, as a direct consequence of the Eells-Sampson Bochner formula for polyhedral domains. Furthermore, our approach enables us to prove rigidity in the borderline case when the eigenvalue of the discrete laplacian is $= \frac{1}{2}$ as conjectured by M.-T. Wang. There are important examples of simplicial complexes associated to p-adic groups where the eigenvalues are indeed equal to $1/2$ [**W2**].

We now turn to the organization of the paper. Throughout the rest of the paper, X will be a compact, admissible and flat simplicial complex and (N, g) a complete Riemannian manifold. Section 2 is a review of the results in [**DM**] about the existence and regularity of harmonic maps. The only difference in this paper is that our maps depend on weights $w(F)$ associated to the n-simplices F of X. This slight modification, which from the point of view of the analysis amounts to taking certain weighted Sobolev spaces associated to the complex, is necessary in order to cover all p-adic buildings. We therefore talk about the w-energy of a map or w-harmonic maps, but this imposes no real analytical difficulty. Section 3 contains our main regularity result and the balancing condition. More precisely, we show:

Theorem (cf. Theorem 4 and Corollary 6). *Let $f : X \to (N^m, g)$ be a w-harmonic map. Then for any n-dimensional simplex F, the restriction of f to $\bar{F} - X^{(n-2)}$ is a $C^{1,\beta}$-map. If $n = 2$, then f is C^∞.*

Theorem (cf. Corollary 5). *Let $f : X \to (N^m, g)$ be a w-harmonic map. For any point $p \in X^{(n-1)} - X^{(n-2)}$, let $F_1, \ldots, F_J$ be the n-simplices containing p and let $E = \cap_{j=1}^J F_j$. Choose coordinates $(x^1, \ldots, x^n)$ on X near p so that E corresponds to the equation $x^n = 0$ and coordinates $(y^1, \ldots, y^m)$ on N near $f(p)$. Set $f_j^\alpha = y^\alpha \circ f|_{F_j}$. Then*

$$\sum_{j=1}^{J} w(F_j) \frac{\partial f_j^\alpha}{\partial x^n}(x^1, \ldots, x^{n-1}, 0) = 0$$

where $w(F_j)$ are the weights associated to F_j.

We also prove:

Theorem (cf. Theorem 8). *Let $f : X \to (N, g)$ be a w-harmonic map where $\dim X = 2$, N has nonpositive sectional curvature and $|\nabla f|$ is a bounded function. Then f is totally geodesic on each simplex of X. Furthermore, if N has negative sectional curvature, then f maps each 2-simplex into a geodesic.*

Finally, in Section 4, we relate the question of regularity of the harmonic map with spectral theory of graphs. In particular, we show:

Theorem (cf. Theorem 12). *Suppose that X is a 2-complex such that every nonzero eigenvalue of the link of every vertex in X satisfies $\lambda \geq 1$. If $f : X \to (N, g)$ is a w-harmonic map into a complete Riemannian manifold of nonpositive sectional curvature, then f is totally geodesic on each 2-simplex of X. In particular, this implies that if the sectional curvature of N is negative, then f maps each simplex into a geodesic. If the eigenvalues satisfy the stronger condition $\lambda > 1$ then f is a constant map.*

We also establish the equivalence of the eigenvalue condition in Theorem 12 with the one appearing in the combinatorial approach. More precisely, we show:

Theorem (cf. Corollary 14). *The condition $\lambda \geq (>) 1$ in the previous theorem is equivalent to the condition that the first nonzero eigenvalue of the discrete Laplacian be $\geq (>) \frac{1}{2}$.*

By taking (Σ^n, c) an arbitrary compact n-dimensional simplical complex, by reducing the weights c to its 2-skeleton $X = \Sigma^{(2)}$ and by applying the previous two theorems on X, we immediately obtain the main theorem in [**W1**].

Corollary (cf. Corollary 15). *Let* (Σ^n, c) *be a compact simplicial complex with admissible weights. Assume that the first nonzero eigenvalue of the link of every vertex is* $> \frac{1}{2}$. *Then* $\pi_1(\Sigma) = \Gamma$ *has property F; i.e., any isometric action of* Γ *on a complete, simply connected manifold* N *of nonpositive sectional curvature has a fixed point on* $\bar{N} = N \cup \partial N$.

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2. Definitions and known results

A simplicial complex of dimension n is referred to as a n-complex. A connected locally finite n-complex is called admissible (cf. [**Ch**] and [**EF**]) if the following two conditions hold:

(i) X is dimensionally homogeneous, i.e., every simplex is contained in a n-simplex, and
(ii) X is locally $(n-1)$-chainable, i.e., for any $(n-2)$-simplex v, every two n-simplices A and B can be joined by a sequence $A = F_0, e_0, F_1, e_1, \ldots, F_{k-1}, e_{k-1}, F_k = B$ where F_i is a n-simplex containing v and e_i is a $(n-1)$-simplex contained in F_i and F_{i+1}.

The boundary ∂X of X is the union of all simplices of dimension $n-1$, which is contained in only one n dimensional simplex. We call a n-complex *flat* if each k-simplex F is isometric to the convex hull of $k+1$ equidistant points of distance 1 in $\mathbf{R}^k$ and every l-simplex L incident to a k-simplex K $(l < k)$ can be seen as a totally geodesic subset of $\bar{K}$. In the sequel, all complexes are admissible, flat, compact and without boundary. An isometric action of a group Γ is a homomorphism $\rho : \Gamma \to \mathrm{Isom}(N)$. Let $\Gamma = \pi_1(X)$. A map $\tilde{\varphi} : \tilde{X} \to N$ is said to be equivariant if

$$\rho(\gamma)\tilde{\varphi}(p) = \tilde{\varphi}(\gamma p)$$

for $\gamma \in \Gamma$ and $p \in X$. If Γ acts freely and properly discontinuously on N, then the map $\tilde{\varphi}$ is a lift of the map $\varphi : X \to N/\Gamma$. By identifying X with a fundamental domain of $\tilde{X}$, we can think of $\tilde{\varphi}$ also being defined on X.

In order to include certain important examples appearing in p-adic geometry (e.g., $p-$adic buildings), we will assume that for each n-dimensional simplex F in X, we have an associated weight $w(F) > 0$ and we define the w-measure $d\mu_w$ by setting

$$d\mu_w = w(F)dx$$

where dx is the standard Lebesgue measure on F. We define the w-energy $E_w(\tilde{\varphi})$ of a map $\tilde{\varphi} : \tilde{X} \to (N, g)$ as

$$E_w(\tilde{\varphi}) = \int_X |\nabla\tilde{\varphi}|^2 d\mu_w = \sum_F w(F) \int_F |\nabla\tilde{\varphi}|^2 dx$$

where $\sum_F$ indicates the sum over all n-dimensional simplices F of X and $|\nabla\tilde{\varphi}|^2$ is defined as usual; i.e.,

$$|\nabla\tilde{\varphi}|^2 = \sum_{k=1}^{n} g\left(\frac{\partial\tilde{\varphi}}{\partial x^k}, \frac{\partial\tilde{\varphi}}{\partial x^k}\right).$$

Of course, if $w(F) = 1$ for all F, then we recover the usual notion of harmonicity defined in [**DM**]. For the sake of notational simplicity, we will fix weights $w(F)$ on F and we will denote $d\mu = d\mu_w$, $E = E_w$, etc. A map $\tilde{f} : \tilde{X} \to N$ is said to be w-harmonic if $E(\tilde{f}) \leq E(\tilde{\varphi})$ for all $W^{1,2}$ equivariant maps $\tilde{\varphi} : \tilde{X} \to N$ (cf. [**EF**]).

The following existence and regularity results for w-harmonic maps from a 2-complex into a non-positively curved Riemannian manifold follows by minor modification of the arguments presented in [**DM**] and [**M**]. (In [**DM**] and [**M**], we only considered weight function w so that $w(F) = 1$ for all 2-simplices F of X.)

Theorem 1. *Let X be a 2-complex with $\Gamma = \pi_1(X)$, N be a complete Riemannian manifold of non-positive curvature and $\rho : \Gamma \to Isom(N)$ be an isometric action of Γ. Assume that ρ does not fix an equivalent class of rays. Then there exists an equivariant w-harmonic map $\tilde{f} : \tilde{X} \to N$.*

Theorem 2. *Let X be a 2-complex, N a complete Riemannian manifold of non-positive curvature and $f : X \to N$ a w-harmonic map. Then f is Lipschitz continuous away from the 0-simplices of X with the Lipschitz bound dependent only on the total w-energy of f and the distance to the 0-simplices. Let p be a 0-simplex and ϱ be the order of f at p. (The definition of order is given in Section 4.) Then there exists $\sigma > 0$ so that*

$$|\nabla f|^2(q) \leq Cr^{2\varrho-2}$$

for all $q \in B_\sigma(p)$ where C depends on $E(f)$ and $r = d_X(p, q)$. More generally, if X is a n-complex and N and f are as above, then f is Lipschitz continuous away from the $(n-2)$-simplices.

3. Regularity results

Let X be a n-complex and N a complete Riemannian manifold as above. For $0 \leq k \leq n$, let $X^{(k)}$ denote the k-skeleton of X. Given $p \in X^{(n-1)} - X^{(n-2)}$, choose an $(n-1)$-simplex E containing p and let $P \in U \mapsto (y^1, \ldots, y^m) \in \mathbf{R}^m$ be a local coordinate system of a neighborhood U of $f(p)$. Assume g is given by $(g_{\alpha\beta})$ in terms of this coordinate system. Choose a neighborhood $V \subset X$ of p sufficiently small so that V does not intersect any $(n-2)$-simplex and $f(V)$ is compactly contained in this coordinate patch. If $F_1, \dots, F_J$ are the n-simplices of X intersecting V and E_ϵ is a ϵ-neighborhood of E, we will use the coordinate system $q \in V \cap (F_j \cup E) \mapsto (x^1, \dots, x^n) \in \mathbf{R}^n$ for $j = 1, \dots, J$ so that a point in E is given by $(x^1, \dots, x^{n-1}, 0)$ and a point in $V \cap E_\epsilon$ is given by $(x^1, \dots, x^n)$ with $0 \leq x^n < \epsilon$. Let E_ϵ be the ϵ-neighborhood of E. Furthermore, for a map $f : V \to N$,

$$f(x^1, \dots, x^n) = (f^1(x^1, \dots, x^n), \dots, f^m(x^1, \dots, x^n))$$

in V and let $f_j^\alpha = f^\alpha|_{F_j}$ for $\alpha = 1, \dots, m$ and $j = 1, \dots, J$.

Theorem 3. *Let $F_1, \dots, F_J, E, E_\epsilon, (x^1, \dots, x^n)$, and $(f^1, \dots, f^m)$ as above. If $f : V \to N$ is a harmonic map, then for any Lipschitz function $\eta : V \to \mathbf{R}$ with compact support and any $\alpha = 1, \dots, m$,*

$$\lim_{\epsilon \to 0} \sum_{j=1}^{J} w(F_j) \int_{F_j \cap \partial E_\epsilon} \eta(x^1, \dots, x^{n-1}, \epsilon) \frac{\partial f_j^\alpha}{\partial x^n} \cdot (x^1, \dots, x^{n-1}, \epsilon) dx^1 \cdots dx^{n-1} = 0.$$

Proof. Let $\varphi = (\varphi^1, \dots, \varphi^m) : V \subset X \to \mathbf{R}^m$ be a Lipschitz continuous map with compact support. For sufficiently small t, define $f_t : V \subset X \to U$ by setting

$$f_t = f + t\varphi = (f^1 + t\varphi^1, \dots, f^m + t\varphi^m).$$

The w-energy of f_t in V is

$$E(f_t; V) = \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_V g_{\alpha\beta}(f(x) + t\varphi(x)) \left(\frac{\partial f^\alpha}{\partial x^k} + t \frac{\partial \varphi^\alpha}{\partial x^k} \right) \left(\frac{\partial f^\beta}{\partial x^k} + t \frac{\partial \varphi^\beta}{\partial x^k} \right) d\mu$$

and since $f = f_0$ is w-energy minimizing,

$$\begin{aligned} 0 &= \frac{d}{dt} E(f_t; V)|_{t=0} \\ &= 2 \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_V g_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \frac{\partial \varphi^\beta}{\partial x^k} d\mu \\ &\quad + \int_V \sum_{k=1}^{n} \sum_{\alpha,\beta,\gamma=1}^{m} g_{\alpha\beta,\gamma}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \frac{\partial f^\beta}{\partial x^k} \varphi^\gamma d\mu \\ &= 2 \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_V \frac{d}{dx^k} \left(g_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \varphi^\beta \right) d\mu \end{aligned}$$

$$
\begin{aligned}
&- 2\sum_{k=1}^{n}\sum_{\alpha,\beta=1}^{m}\int_V g_{\alpha\beta}(f(x))\frac{\partial^2 f^\alpha}{\partial(x^k)^2}\varphi^\beta d\mu \\
&- 2\sum_{k=1}^{n}\sum_{\alpha,\beta,\gamma=1}^{m}\int_V g_{\alpha\beta,\gamma}(f(x))\frac{\partial f^\alpha}{\partial x^k}\frac{\partial f^\gamma}{\partial x^k}\varphi^\beta d\mu \\
&+ \sum_{k=1}^{n}\sum_{\alpha,\beta,\gamma=1}^{m}\int_V g_{\alpha\beta,\gamma}(f(x))\frac{\partial f^\alpha}{\partial x^k}\frac{\partial f^\beta}{\partial x^k}\varphi^\gamma d\mu \\
= {}& 2\sum_{k=1}^{n}\sum_{\alpha,\beta=1}^{m}\int_V \frac{d}{dx^k}\left(g_{\alpha\beta}(f(x))\frac{\partial f^\alpha}{\partial x^k}\varphi^\beta\right) d\mu \\
&- 2\sum_{k=1}^{n}\sum_{\alpha,\beta=1}^{m}\int_V g_{\alpha\beta}(f(x))\frac{\partial^2 f^\alpha}{\partial(x^k)^2}\varphi^\beta d\mu \\
&- \sum_{k=1}^{n}\sum_{\alpha,\beta,\gamma=1}^{m}\int_V g_{\alpha\beta,\gamma}(f(x))\frac{\partial f^\alpha}{\partial x^k}\frac{\partial f^\gamma}{\partial x^k}\varphi^\beta d\mu \\
&+ \sum_{k=1}^{n}\sum_{\alpha,\beta=1}^{m}\int_V g_{\gamma\beta,\alpha}(f(x))\frac{\partial f^\alpha}{\partial x^k}\frac{\partial f^\gamma}{\partial x^k}\varphi^\beta d\mu \\
&+ \sum_{k=1}^{n}\sum_{\alpha,\beta,\gamma=1}^{m}\int_V g_{\alpha\beta,\gamma}(f(x))\frac{\partial f^\alpha}{\partial x^k}\frac{\partial f^\beta}{\partial x^k}\varphi^\gamma d\mu \\
= {}& 2\sum_{k=1}^{n}\sum_{\alpha,\beta=1}^{m}\int_V \frac{d}{dx^k}\left(g_{\alpha\beta}(f(x))\frac{\partial f^\alpha}{\partial x^k}\varphi^\beta\right) d\mu \\
&- 2\sum_{k=1}^{n}\sum_{\alpha,\beta=1}^{m}\int_V g_{\alpha\beta}(f(x))\frac{\partial^2 f^\alpha}{\partial(x^k)^2}\varphi^\beta d\mu \\
&- \sum_{k=1}^{n}\sum_{\alpha,\beta,\gamma=1}^{m}\int_V (g_{\alpha\beta,\gamma}+g_{\gamma\beta,\alpha}-g_{\alpha\gamma,\beta})\frac{\partial f^\alpha}{\partial x^k}\frac{\partial f^\gamma}{\partial x^k}\varphi^\beta d\mu.
\end{aligned}
$$

Let $\eta_\alpha = \sum_{\beta=1}^m g_{\alpha\beta}\varphi^\beta$. Then $\varphi^\beta = \sum_{\alpha=1}^m g^{\alpha\beta}\eta_\alpha$ and we see that the last two terms above equal

$$
\begin{aligned}
&-\int_V \sum_{\alpha=1}^{m} 2\triangle f^\alpha \eta_\alpha + \sum_{k=1}^{n}\sum_{\alpha,\beta,\gamma,\delta=1}^{m} g^{\delta\beta}(g_{\alpha\beta,\gamma}+g_{\gamma\beta,\alpha}-g_{\alpha\gamma,\beta})\frac{\partial f^\alpha}{\partial x^k}\frac{\partial f^\gamma}{\partial x^k}\eta_\delta d\mu \\
&= -2\int_V \left(\sum_{\alpha=1}^{m}\triangle f^\alpha\eta_\alpha + 2\sum_{k=1}^{n}\sum_{\alpha,\gamma,\delta=1}^{m}\Gamma^\delta_{\alpha\gamma}(f(x))\frac{\partial f^\alpha}{\partial x^k}\frac{\partial f^\gamma}{\partial x^k}\eta_\delta\right) d\mu = 0.
\end{aligned}
$$

The last equality is because f is a smooth harmonic map in the interior of each n-simplex F and we have the pointwise equality,

$$\triangle f^\alpha + \sum_{k=1}^{n} \sum_{\beta,\gamma=1}^{m} \Gamma^\alpha_{\beta\gamma}(f(x)) \frac{\partial f^\beta}{\partial x^k} \frac{\partial f^\gamma}{\partial x^k} = 0$$

in $F_j \cap V$. Therefore, by the monotone convergence theorem and the fact that f is Lipschitz away from the $(n-2)$-skeleton by Theorem 2, we conclude

$$\begin{aligned} 0 &= \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_V \frac{d}{dx^k} \left(g_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \varphi^\beta(x) \right) d\mu \\ &= \lim_{\epsilon \to 0} \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_{V - E_\epsilon} \frac{d}{dx^k} \left(g_{\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^k} \varphi^\beta(x) \right) d\mu \\ &= \lim_{\epsilon \to 0} \sum_{j=1}^{J} w(F_j) \sum_{k=1}^{n} \sum_{\alpha,\beta=1}^{m} \int_{F_j - E_\epsilon} \frac{d}{dx^k} \left(g_{\alpha\beta}(f_j(x)) \frac{\partial f_j^\alpha}{\partial x^k} \varphi^\beta(x) \right) dx. \end{aligned}$$

Since φ^β has compact support in V,

$$\sum_{\alpha,\beta=1}^{m} \int_{F_j - E_\epsilon} \frac{d}{\partial x^k} \left(g_{\alpha\beta}(f_j(x)) \frac{\partial f_j^\alpha}{\partial x^k} \varphi^\beta(x) \right) dx = 0$$

for $k = 1, \ldots, n-1$ and $j = 1, \ldots, J$. Hence,

$$\begin{aligned} 0 &= \lim_{\epsilon \to 0} \sum_{j=1}^{J} w(F_j) \sum_{\alpha,\beta=1}^{m} \int_{F_j - E_\epsilon} \frac{d}{dx^n} \left(g_{\alpha\beta}(f_j(x)) \frac{\partial f_j^\alpha}{\partial x^k} \varphi^\beta(x) \right) dx \\ &= \lim_{\epsilon \to 0} \sum_{j=1}^{J} w(F_j) \sum_{\alpha,\beta=1}^{m} \int_{F_j \cap \partial E_\epsilon} g_{\alpha\beta}(f_j(x^1, \ldots, x^{n-1}, \epsilon)) \frac{\partial f_j^\alpha}{\partial x^n} \\ &\quad \cdot \varphi^\beta(x^1, \ldots, x^{n-1}, \epsilon) dx. \end{aligned}$$

Let $\eta : V \to \mathbf{R}$ be a Lipschitz continuous compactly supported function and $\varphi^\beta = g^{\beta\alpha_0}\eta$ for $\alpha_0, \beta \in \{1, \ldots, m\}$. Then

$$\begin{aligned} \sum_{\alpha,\beta=1}^{m} g_{\alpha\beta} \frac{\partial f_j^\alpha}{\partial x^n} \varphi^\beta &= \sum_{\alpha,\beta=1}^{m} g_{\alpha\beta} g^{\beta\alpha_0} \eta \frac{\partial f_j^\alpha}{\partial x^n} \\ &= \sum_{\alpha=1}^{m} \delta_{\alpha\alpha_0} \eta \frac{\partial f_j^\alpha}{\partial x^n} \\ &= \eta \frac{\partial f_j^{\alpha_0}}{\partial x^n} \end{aligned}$$

and therefore

$$\lim_{\epsilon\to 0}\sum_{j=1}^{J} w(F_j)\int_{\partial E_\epsilon}\eta(x^1,\dots,x^{n-1},\epsilon)\frac{\partial f_j^{\alpha_0}}{\partial x^n}(x^1,\dots,x^{n-1},\epsilon)dx = 0.$$

q.e.d.

We now prove our main regularity result.

Theorem 4. *Let $f : V \subset X \to (N^n, g)$ be a w-harmonic map so that $f(V)$ maps into a coordinate neighborhood. For any point $p \in V \cap X^{(n-1)} - X^{(n-2)}$, let E be the $(n-1)$-simplex containing p and $F_1,\dots,F_J$ be the n-simplices containing E in its closure. As in Theorem 3, let f_j^α be the restriction of the coordinate function f^α to the n-simplex F_j and $(x^1,\dots,x^n)$ be a coordinate system for V. Then there exist a neighborhood $\Omega \subset\subset V$ of p so that $f_j^\alpha \in C^{1,\beta}(\Omega\cap\overline{F_j})$, $f_j^\alpha \in W^{2,2}(\Omega\cap F_j)$ and $\frac{\partial f_j^\alpha}{\partial x^k} \in W^{2,2}(\Omega\cap F_j)$ for $k = 1,\dots,n-1$.*

Proof. Assume p corresponds to $(0,\dots,0)$, let

$$w_j = w(F_j)/(w(F_1)+\dots+w(F_J))$$

and $B_r \subset \mathbf{R}^n$ be a ball of radius r centered at the origin. Fix $\alpha_0 \in \{1,\dots,m\}$ and $j_0 \in \{1,\dots,J\}$ and, for $r > 0$ sufficiently small, define $\psi : B_r \to \mathbf{R}$ by setting

$$\psi(\hat{x},x^n) = \begin{cases} f_{j_0}^{\alpha_0}(\hat{x},x^n) & \text{when } x^n \geq 0\\ -f_{j_0}^{\alpha_0}(\hat{x},-x^n) + \frac{2}{J}\sum_{j=1}^{J} w_j f_j^{\alpha_0}(\hat{x},-x^n) & \text{when } x^n < 0\end{cases}$$

where we simplify notation by using $(\hat{x},x^n) = (x^1,\dots,x^{(n-1)},x^n)$. By the (Lipschitz) continuity of f, $f_j^{\alpha_0}(\hat{x},0) = f_{j_0}^{\alpha_0}(\hat{x},0)$ for all $j = 1,\dots,J$. Therefore,

$$\begin{aligned} -f_{j_0}^{\alpha_0}(\hat{x},0) + \frac{2}{J}\sum_{j=1}^{J} w_j f_j^{\alpha_0}(\hat{x},0) &= -f_{j_0}^{\alpha_0}(\hat{x},0) + \frac{2}{J} f_{j_0}^{\alpha_0}(\hat{x},0)\sum_{j=1}^{J} w_j \\ &= f_{j_0}^{\alpha_0}(\hat{x},0). \end{aligned}$$

This implies that ψ is Lipschitz continuous in B_r. Additionally, for fixed $\epsilon > 0$,

$$\frac{\partial\psi}{\partial x^n}(\hat{x},-\epsilon) = \frac{\partial f_{j_0}^{\alpha_0}}{\partial x^n}(\hat{x},\epsilon) - \frac{2}{J}\sum_{j=1}^{J} w_j\frac{\partial f_j^{\alpha_0}}{\partial x^n}(\hat{x},\epsilon).$$

Since

$$\lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=\epsilon\}}\sum_{j=1}^{J}\varphi(\hat{x},-\epsilon)w_j\frac{\partial f_j^{\alpha_0}}{\partial x^n}(\hat{x},\epsilon) = 0$$

for any $\varphi \in C_c^\infty(B_r)$ by Theorem 3, we see that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=-\epsilon\}} \varphi(\hat{x},-\epsilon)\frac{\partial\psi}{\partial x^n}(\hat{x},-\epsilon)\\
&=\lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=-\epsilon\}} \varphi(\hat{x},-\epsilon)\frac{\partial f_{j_0}^{\alpha_0}}{\partial x^n}(\hat{x},\epsilon)\\
&\quad -2\lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=-\epsilon\}} \sum_{j=1}^{J}\varphi(\hat{x},-\epsilon)w_j\frac{\partial f_{j}^{\alpha_0}}{\partial x^n}(\hat{x},\epsilon)\\
&=\lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=-\epsilon\}} \varphi(\hat{x},-\epsilon)\frac{\partial f_{j_0}^{\alpha_0}}{\partial x^n}(\hat{x},\epsilon)\\
&=\lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=\epsilon\}} \varphi(\hat{x},\epsilon)\frac{\partial f_{j_0}^{\alpha_0}}{\partial x^n}(\hat{x},\epsilon)\\
&\quad +\lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=\epsilon\}} (\varphi(\hat{x},-\epsilon)-\varphi(\hat{x},\epsilon))\frac{\partial f_{j_0}^{\alpha_0}}{\partial x^n}(\hat{x},\epsilon)\\
&=\lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=\epsilon\}} \varphi(\hat{x},\epsilon)\frac{\partial \psi}{\partial x^n}(\hat{x},\epsilon)\\
&\quad +\lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=\epsilon\}} (\varphi(\hat{x},-\epsilon)-\varphi(\hat{x},\epsilon))\frac{\partial f_{j_0}^{\alpha_0}}{\partial x^n}(\hat{x},\epsilon).
\end{aligned}$$

The last term is equal to zero by the dominated convergence theorem since $\varphi(\hat{x},-\epsilon)-\varphi(\hat{x},\epsilon)$ converges uniformly to 0 as $\epsilon\to 0$ and $\frac{\partial f_{j_0}^{\alpha_0}}{\partial x^n}$ is bounded. Thus,

$$\lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=-\epsilon\}} \varphi\frac{\partial\psi}{\partial x^n} = \lim_{\epsilon\to 0}\int_{B_r\cap\{x^n=\epsilon\}} \varphi\frac{\partial\psi}{\partial x^n}. \tag{3}$$

Also, since f_j is a smooth harmonic map in the interior of a n-simplex,

$$\triangle f_j^\alpha + \sum_{l=1}^{n}\sum_{\beta,\gamma=1}^{m}\Gamma_{\beta\gamma}^\alpha(f_j(x))\frac{\partial f_j^\beta}{\partial x^l}\frac{\partial f_j^\gamma}{\partial x^l}=0. \tag{4}$$

Let

$$\Gamma_j^\alpha(\hat{x},x^n)=\sum_{l=1}^{n}\sum_{\beta,\gamma=1}^{m}\Gamma_{\beta\gamma}^\alpha(f_j(\hat{x},x^n))\frac{\partial f_j^\beta}{\partial x^l}(\hat{x},x^n)\frac{\partial f_j^\gamma}{\partial x^l}(\hat{x},x^n)$$

and $F : B_r \to \mathbf{R}$ be defined by

$$F(\hat{x},x^n)=\begin{cases}\Gamma_{j_0}^{\alpha_0}(\hat{x},x^n) & \text{when } x^n\geq 0\\ -\Gamma_{j_0}^{\alpha_0}(\hat{x},-x^n)+2\sum_{j=1}^{J}w_j\Gamma_j^{\alpha_0}(\hat{x},-x^n) & \text{when } x^n<0.\end{cases}$$

The equality (4) implies that $\triangle\psi(\hat{x},x^n) = -F(\hat{x},x^n)$ for $x^n > 0$ and $x^n < 0$. Thus, integration by parts implies

$$\int_{B_r\cap\{x^n>\epsilon\}} \nabla\varphi\cdot\nabla\psi = -\int_{B_r\cap\{x^n>\epsilon\}} \varphi\triangle\psi + \int_{B_r\cap\{x^n=\epsilon\}} \varphi\frac{\partial\psi}{\partial x^n}$$
$$= \int_{B_r\cap\{x^n>\epsilon\}} \varphi F - \int_{B_r\cap\{x^n=\epsilon\}} \varphi\frac{\partial\psi}{\partial x^n}$$

and

$$\int_{B_r\cap\{x^n<-\epsilon\}} \nabla\varphi\cdot\nabla\psi = -\int_{B_r\cap\{x^n<-\epsilon\}} \varphi\triangle\psi - \int_{B_r\cap\{x^n=-\epsilon\}} \varphi\frac{\partial\psi}{\partial x^n}$$
$$= \int_{B_r\cap\{x^n<-\epsilon\}} \varphi F + \int_{B_r\cap\{x^n=-\epsilon\}} \varphi\frac{\partial\psi}{\partial x^n}$$

for any $\varphi \in C_c^\infty(B_r)$. Summing the above two equalities, letting $\epsilon \to 0$ and noting (3), we see that

$$\int_{B_r} \nabla\varphi\cdot\nabla\psi = \int_{B_r} \varphi F.$$

Since F is bounded, the standard elliptic regularity theorem implies that $\psi \in C^{1,\beta}(B_{r'})$ and $\psi \in W^{2,2}(B_{r'})$ for $r' < r$. Since $\psi(\hat{x},x^n) = f_{j_0}^{\alpha_0}(\hat{x},x^n)$ for $x^n \geq 0$ and $\alpha_0 \in \{1,\dots,m\}$, $j_0 \in \{1,\dots,J\}$ are arbitrary choices, this shows $f_j^\alpha \in C^{1,\beta}(B_{r'}\cap\{x^n \geq 0\})$ and $f_j^\alpha \in W^{2,2}(B_{r'}\cap\{x^n > 0\})$ for any $\alpha = 1,\dots,n$ and $j = 1,\dots,J$. Thus, the fact that $\frac{\partial^2 f_j^\alpha}{\partial x^l\partial x^k} \in L^2(B_{r'}\cap\{x^n > 0\})$ combined with $\frac{\partial f_j^\alpha}{\partial x^l}$ being bounded implies that

$$\frac{\partial}{\partial x^k}\left(\sum_{l=1}^n\sum_{\beta,\gamma=1}^m \Gamma^\alpha_{\beta\gamma}(f(x))\frac{\partial f^\beta}{\partial x^l}\frac{\partial f^\gamma}{\partial x^l}\right)$$
$$= \sum_{l=1}^n\sum_{\beta,\gamma=1}^m \Gamma^\alpha_{\beta\gamma,\delta}(f(x))\frac{\partial f^\beta}{\partial x^l}\frac{\partial f^\gamma}{\partial x^l}\frac{\partial f^\delta}{\partial x^k}$$
$$+2\sum_{l=1}^n\sum_{\beta,\gamma=1}^m \Gamma^\alpha_{\beta\gamma}(f(x))\frac{\partial^2 f^\beta}{\partial x^l\partial x^k}\frac{\partial f^\gamma}{\partial x^l}$$

is $L^2(B_{r'}\cap\{x^n > 0\})$. This implies $\frac{\partial F}{\partial x^k}$ is $L^2(B_{r'})$. Now let $k = 1,\dots,n-1$. Then

$$\int_{B_r\cap\{x^n>\epsilon\}} \nabla\varphi\cdot\nabla\frac{\partial\psi}{\partial x^k} = -\int_{B_r\cap\{x^n>\epsilon\}} \varphi\triangle\frac{\partial\psi}{\partial x^k} + \int_{B_r\cap\{x^n=\epsilon\}} \varphi\frac{\partial^2\psi}{\partial x^k\partial x^n}$$
$$= \int_{B_r\cap\{x^n>\epsilon\}} \varphi\frac{\partial F}{\partial x^k} + \int_{B_r\cap\{x^n=\epsilon\}} \varphi\frac{\partial^2\psi}{\partial x^k\partial x^n}$$
$$= \int_{B_r\cap\{x^n>\epsilon\}} \varphi\frac{\partial F}{\partial x^k} - \int_{B_r\cap\{x^n=\epsilon\}} \frac{\partial\varphi}{\partial x^k}\frac{\partial\psi}{\partial x^n}$$

and

$$\begin{aligned}\int_{B_r\cap\{x^n<-\epsilon\}}\nabla\varphi\cdot\nabla\frac{\partial\psi}{\partial x^k} &= -\int_{B_r\cap\{x^n<-\epsilon\}}\varphi\triangle\frac{\partial\psi}{\partial x^k}-\int_{B_r\cap\{x^n=-\epsilon\}}\varphi\frac{\partial^2\psi}{\partial x^k\partial x^n}\\ &= \int_{B_r\cap\{x^n<-\epsilon\}}\varphi\frac{\partial F}{\partial x^k}-\int_{B_r\cap\{x^n=-\epsilon\}}\varphi\frac{\partial^2\psi}{\partial x^k\partial x^n}\\ &= \int_{B_r\cap\{x^n<-\epsilon\}}\varphi\frac{\partial F}{\partial x^k}+\int_{B_r\cap\{x^n=-\epsilon\}}\frac{\partial\varphi}{\partial x^k}\frac{\partial\psi}{\partial x^n}.\end{aligned}$$

Adding the above two equalities, letting $\epsilon\to 0$ and using (3) with φ replaced by $\frac{\partial\varphi}{\partial x^k}$, we obtain

$$\int_{B_r}\nabla\varphi\cdot\nabla\frac{\partial\psi}{\partial x^k}=\int_{B_r}\varphi\frac{\partial F}{\partial x^k}.$$

Since $\frac{\partial F}{\partial x^k}\in L^2(B_{r'})$, we conclude $\frac{\partial\psi}{\partial x^k}\in W^{2,2}(B_{r''})$ for $r''<r'$. Thus, $\frac{\partial f_j^\alpha}{\partial x^k}$ is $W^{2,2}(B_{r''}\cap\{x^n>0\})$. q.e.d.

Corollary 5. *Let $V,F_1,\dots,F_J,E,(x^1,\dots,x^n),(f^1,\dots,f^m),f_j^\alpha$ be as in the paragraph proceeding Theorem 3. If $f:V\to N$ is a harmonic map, then*

$$\sum_{j=1}^J w(F_j)\frac{\partial f_j^\alpha}{\partial x^n}(x^1,\dots,x^{n-1},0)=0$$

for all $\alpha=1,\dots,m$ and all $(x^1,\dots,x^{n-1},0)\in V\cap E$.

Proof. This pointwise equality follows immediately from Theorem 3 and the regularity result of Theorem 4. q.e.d.

The following corollary to Theorem 4 will be important in Theorem 7 when we restrict our attention to two-dimensional domains.

Corollary 6. *Assume $n=2$. Then for every 2-simplex F, the restriction of f^α to the closure of F is C^∞ away from the 0-simplices.*

Proof. We can push the argument for the proof of Theorem 4 further in the case $n=2$. We let $x=x^1$ and $y=x^2$. Using the Lipschitz continuity to bound the first partial derivatives of f, we see that

$$\begin{aligned}&\left|\frac{\partial^2}{\partial x^2}\left(\sum_{\beta,\gamma=1}^m\Gamma^\alpha_{\beta\gamma}(f(x))\left(\frac{\partial f^\beta}{\partial x}\frac{\partial f^\gamma}{\partial x}+\frac{\partial f^\beta}{\partial y}\frac{\partial f^\gamma}{\partial y}\right)\right)\right|\\ &\le\ C_0+C_1|D^2f|+C_2|D^2f|^2+C_3|D^3f|\end{aligned}$$

where C_0,C_1,C_2,C_3 are constants, and $|D^if|$ is the sum of the absolute values of the ith order partial derivatives of f that involves at most one $\frac{\partial}{\partial y}$. By Theorem 4, $\frac{\partial f_j^\alpha}{\partial x}\in W^{2,2}(B_{r_1}^+)$ for some $r_1>0$ where we use the notation $B_r^+=B_r\cap\{y\ge 0\}$. Thus, $\frac{\partial^2 f_j^\alpha}{\partial x^2},\frac{\partial^2 f_j^\alpha}{\partial x\partial y}\in W^{1,2}(B_{r_1})$ and hence the Sobolev embedding theorem implies that $\frac{\partial^2 f_j^\alpha}{\partial x^2}$, $\frac{\partial^2 f_j^\alpha}{\partial x \partial y} \in L^q(B_{r_1})$ for any $q < \infty$. Therefore, we conclude that $\frac{\partial^2 F}{\partial x^2} \in L^2(B_{r'})$. Using an analogous argument as in the proof of Theorem 4, we can show that

$$\int_{B_{r_1}} \nabla\varphi \cdot \nabla \frac{\partial^2 \psi}{\partial x^2} = \int_{B_{r_1}} \varphi \frac{\partial^2 F}{\partial x^2}$$

for any $\varphi \in C_c^\infty(B_{r_1})$. Thus, $\frac{\partial^2 \psi}{\partial x^2} \in W^{2,2}(B_{r_2})$ for some $r_2 < r_1$ which immediately implies $\frac{\partial^2 f_j^\alpha}{\partial x^2} \in W^{2,2}(B^+_{r_2})$.

Now we continue inductively. Assume $\frac{\partial^l f_j^\alpha}{\partial x^l} \in W^{2,2}(B^+_{r_l})$. Then $\frac{\partial^{l+1} f_j^\alpha}{\partial x^{l+1}}, \frac{\partial^{l+1} f_j^\alpha}{\partial x^l \partial y} \in W^{1,2}(B^+_{r_l})$ and $\frac{\partial^{l+1} f_j^\alpha}{\partial x^{l+1}}, \frac{\partial^{l+1} f_j^\alpha}{\partial x^l \partial y} \in L^q(B^+_{r_l})$ for any $q = 1, 2, \ldots$. Then $\frac{\partial^{l+1} F}{\partial x^{l+1}} \in L^2(B^+_{r_l})$ since

$$\begin{aligned} &\left| \frac{\partial^{l+1}}{\partial x^{l+1}} \left(\sum_{\beta,\gamma=1}^m \Gamma^\alpha_{\beta\gamma}(f(x)) \left(\frac{\partial f^\beta}{\partial x}\frac{\partial f^\gamma}{\partial x} + \frac{\partial f^\beta}{\partial y}\frac{\partial f^\gamma}{\partial y} \right) \right) \right| \\ &\quad \leq \; P(|D^2 f|, \ldots, |D^{l+1} f|) + C_{l+2}|D^{l+2} f| \end{aligned}$$

where C_{l+1} is a constant, and $|D^i f|$ is the sum of absolute values of the ith order partial derivatives of f that involves at most one $\frac{\partial}{\partial y}$ and $P(X_2, \ldots, X_{l+1})$ is a polynomial involving variables $X_2, \ldots, X_{l+1}$. Using the weak differential equality

$$\int_{B_{r_l}} \nabla\varphi \cdot \nabla \frac{\partial^{l+1}\psi}{\partial x^{l+1}} = \int_{B_{r_l}} \varphi \frac{\partial^{l+1} F}{\partial x^{l+1}}, \ \forall \varphi \in C_c^\infty(B_{r_l})$$

we see that $\frac{\partial^{l+1} f_j^\alpha}{\partial x^{l+1}} \in W^{2,2}(B^+_{r_{l+1}})$ for $r_{l+1} < r_l$.

We now show that f_j^α is C^∞. By the trace theory, $\frac{\partial^{l+1} f_j^\alpha}{\partial x^{l+1}} \in W^{1,2}(B_{r_{l+1}} \cap \{y = 0\})$ and hence $f_j^\alpha \in W^{l+2,2}(B_{r_{l+1}} \cap \{y = 0\})$ which in turn implies $f_j^\alpha \in C^{l+1}(B_{r_{l+1}} \cap \{y = 0\})$. Since f_j^α is a smooth harmonic map in $B_{r_{l+1}} \cap \{y > 0\}$, this implies that $f_j^\alpha \in C^l(B^+_{r_{l+1}})$ by the boundary regularity theory of harmonic maps. Since l can be made arbitrarily large, f_j^α is C^∞. q.e.d.

For a 2-complex X, the balancing condition along the 1-simplex stated in Corollary 5 implies that Stoke's theorem can be applied to the integral of $\triangle|\nabla f|^2$ as long as $|\nabla f|$ is a bounded function in X.

Theorem 7. *Let X be a 2-complex and $f : \tilde{X} \to (N, g)$ be a w-harmonic map. If there exists C so that $|\nabla f| \leq C$, then*

$$\int_X \triangle|\nabla f|^2 d\mu = 0.$$

Proof. The Bochner formula gives

$$\frac{1}{2}\triangle|\nabla f|^2(x,y) = |\nabla df|^2 - \left\langle R^N\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right\rangle$$

where $R^N(\cdot,\cdot) \leq 0$ by hypothesis. In particular, $\triangle|\nabla f|^2 \geq 0$ and the monotone convergence theorem implies

$$\begin{aligned} 0 &\leq \int_X \triangle|\nabla f|^2 d\mu \\ &= \lim_{\sigma\to 0}\int_{X-\cup_{v\in\mathcal{V}}B_\sigma(v)} \triangle|\nabla f|^2 d\mu. \end{aligned} \tag{5}$$

Now applying Stoke's theorem to each 2-simplex F of X, we have

$$0 \leq \int_X \triangle|\nabla f|^2 d\mu \tag{6}$$

$$\begin{aligned} &= \lim_{\sigma\to 0}\sum_F w(F)\int_{\partial F-\cup_{v\in\mathcal{V}_F}B_\sigma(v)} \frac{\partial}{\partial\eta}|\nabla f|^2 ds \\ &\quad + \lim_{\sigma\to 0}\sum_F w(F)\sum_v\int_{\partial B_\sigma(v)\cap F}\frac{\partial}{\partial\eta}|\nabla f|^2 ds \end{aligned} \tag{7}$$

where $\sum_F$ and $\sum_v$ indicates the sum over all 2-simplices F and 0-simplices v of X respectively and η is the outward pointing normal to $F - \cup_{v\in\mathcal{V}_F}B_\sigma(v)$. For an edge point $p \in X$, let $(x^1, x^2) = (x, y)$, $(f^1, \ldots, f^m)$, $F_1, \ldots, F_J$ and f_j^α be as in the paragraph proceeding Theorem 3. Then for every $(x, 0) \in V$

$$\begin{aligned} &\sum_{j=1}^J w(F_j)\frac{\partial}{\partial\eta}|\nabla f_j|^2(x,0) \\ &= -\sum_{j=1}^J w(F_j)\frac{\partial}{\partial y}|\nabla f_j|^2(x,0) \\ &= -\sum_{j=1}^J w(F_j)\frac{\partial}{\partial y} \\ &\qquad \cdot\left(\sum_{\alpha,\beta=1}^m g_{\alpha\beta}(f_j(x,y))\left(\frac{\partial f_j^\alpha}{\partial x}\frac{\partial f_j^\beta}{\partial x} + \frac{\partial f_j^\alpha}{\partial y}\frac{\partial f_j^\beta}{\partial y}\right)\right)\Big|_{(x,y)=(x,0)}. \end{aligned}$$

Now recall that the function $\phi_j : F_j \to \mathbf{C}$ defined by

$$\phi_j(x,y) = \sum_{\alpha,\beta=1}^n g_{\alpha\beta}(f_j(x,y))\left(\frac{\partial f_j^\alpha}{\partial x}\frac{\partial f_j^\beta}{\partial x} - \frac{\partial f_j^\alpha}{\partial y}\frac{\partial f_j^\beta}{\partial y} - 2i\frac{\partial f_j^\alpha}{\partial x}\frac{\partial f_j^\beta}{\partial y}\right)(x,y)$$

is a holomorphic function [**S**]. By Corollary 5,

$$\begin{aligned}
&\mathrm{Im}\sum_{j=1}^{J} w(F_j)\phi_j(x,0)\\
&\quad = \quad 2\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(f_1(x,0))\frac{\partial f_1^\alpha}{\partial x}(x,0)\sum_{j=1}^{J} w(F_j)\frac{\partial f_j^\beta}{\partial y}(x,0)=0.
\end{aligned}$$

Here, we have used the fact that f is smooth and hence $f_j^\alpha(x,0) = f_1^\alpha(x,0)$ and $\frac{\partial f_j^\alpha}{\partial x}(x,0) = \frac{\partial f_1^\alpha}{\partial x}(x,0)$ for each $j = 1,\dots,J$. Hence

$$\phi(x,y) = \sum_{j=1}^{J} w(F_j)\phi_j(x,y)$$

extends across $y = 0$ as a holomorphic function. By the Cauchy-Riemann equation,

$$\begin{aligned}
0 &= -\left(\frac{\partial}{\partial x}\mathrm{Im}\phi\right)(x,0)\\
&= \left(\frac{\partial}{\partial y}\mathrm{Re}\phi\right)(x,0)\\
&= \sum_{j=1}^{J} w(F_j)\frac{\partial}{\partial y}\\
&\quad \cdot \left(\sum_{\alpha,\beta=1}^{m} g_{\alpha\beta}(f_j(x,y))\left(\frac{\partial f_j^\alpha}{\partial x}\frac{\partial f_j^\beta}{\partial x} - \frac{\partial f_j^\alpha}{\partial y}\frac{\partial f_j^\beta}{\partial y}\right)(x,y)\right)\Big|_{(x,y)=(x,0)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{j=1}^{J} w(F_j)\frac{\partial}{\partial y}|\nabla f_j|^2(x,0)\\
&= 2\sum_{j=1}^{J} w(F_j)\frac{\partial}{\partial y}\left(\sum_{\alpha,\beta=1}^{m} g_{\alpha\beta}(f_j(x,y))\frac{\partial f_j^\alpha}{\partial x}\frac{\partial f_j^\beta}{\partial x}(x,y)\right)\Big|_{(x,y)=(x,0)}\\
&= 2\sum_{j=1}^{J} w(F_j)\sum_{\alpha,\beta=1}^{m} g_{\alpha\beta,\gamma}(f_j(x,0))\frac{\partial f_j^\alpha}{\partial x}\frac{\partial f_j^\beta}{\partial x}\frac{\partial f_j^\gamma}{\partial y}(x,0)\\
&\quad + 4\sum_{j=1}^{J} w(F_j)\sum_{\alpha,\beta=1}^{m} g_{\alpha\beta}(f_j(x,0))\frac{\partial f_j^\alpha}{\partial x}\frac{\partial^2 f_j^\beta}{\partial x\partial y}(x,0).
\end{aligned}$$

Again, using the fact that f is smooth, which implies $f_j(x,0) = f_1(x,0)$ and $\frac{\partial f_j}{\partial x}(x,0) = \frac{\partial f_1}{\partial x}(x,0)$, we have

$$\begin{aligned}
&\sum_{j=1}^{J} w(F_j)\frac{\partial}{\partial y}|\nabla f_j|^2(x,0) \\
&= 2\sum_{\alpha,\beta=1}^{m} g_{\alpha\beta,\gamma}(f_j(x,0))\frac{\partial f_1^\alpha}{\partial x}\frac{\partial f_1^\beta}{\partial x}(x,0)\left(\sum_{j=1}^{J} w(F_j)\frac{\partial f_j^\gamma}{\partial y}(x,0)\right) \\
&\quad + 4\sum_{\alpha,\beta=1}^{m} g_{\alpha\beta}(f_j(x,0))\frac{\partial f_1^\alpha}{\partial x}(x,0)\frac{\partial}{\partial x}\left(\sum_{j=1}^{J} w(F_j)\frac{\partial f_j^\beta}{\partial y}\right)(x,0).
\end{aligned}$$

By Corollary 5,

$$\sum_{j=1}^{J} w(F_j)\frac{\partial f_j^\gamma}{\partial y}(x,0) = 0.$$

Thus, we conclude

$$\sum_F w(F)\int_{\partial F - \cup_{v\in\mathcal{V}} B_\sigma(v)} \frac{\partial}{\partial\eta}|\nabla f|^2 ds = 0 \tag{8}$$

for any $\sigma > 0$. This implies

$$0 \leq \int_{X - \cup_{v\in\mathcal{V}} B_\sigma(v)} \triangle|\nabla f|^2 d\mu = \sum_F w(F)\sum_v \int_{\partial B_\sigma(v)\cap F} \frac{\partial}{\partial\eta}|\nabla f|^2 ds \tag{9}$$

by equation (5). With r denoting the distance from the vertex v,

$$\int_{\partial B_\sigma(v)\cap F} \frac{\partial}{\partial r}|\nabla f|^2 ds = \sigma\frac{d}{d\sigma}\left(\frac{1}{\sigma}\int_{\partial B_\sigma(v)\cap F}|\nabla f|^2 ds\right).$$

Since $\frac{\partial}{\partial\eta} = -\frac{\partial}{\partial r}$, equation (9) implies

$$\frac{d}{d\sigma}\left(\sum_F w(F)\sum_v \frac{1}{\sigma}\int_{\partial B_\sigma(v)\cap F}|\nabla f|^2 ds\right) \leq 0$$

and

$$\sigma \mapsto \sum_F w(F)\sum_v \frac{1}{\sigma}\int_{\partial B_\sigma(v)\cap F}|\nabla f|^2 ds$$

is monotone non-increasing. On the other hand, the hypothesis implies that

$$\frac{1}{\sigma}\int_{\partial B_\sigma(v)\cap F}|\nabla f|^2 ds \leq C'$$

for some constant C'. Thus,

$$L = \lim_{\sigma\to 0}\sum_F w(F)\sum_v \frac{1}{\sigma}\int_{\partial B_\sigma(v)\cap F}|\nabla f|^2 ds$$

exists. Let

$$G(\epsilon) = \begin{cases} 0 & \text{for } \epsilon = 0 \\ \sum_F w(F) \sum_v \int_{\partial B_\epsilon(v)\cap F} |\nabla f|^2 ds & \text{for } \epsilon \in (0, \frac{1}{2}]. \end{cases}$$

Since $|\nabla f|^2$ is C^1 away from the vertices, G is a differentiable function on $(0, \frac{1}{2})$. Furthermore, since $|\nabla f|$ is bounded, G is continuous on $[0, \frac{1}{2}]$. Therefore, for $\sigma_i \to 0$, we can choose $\epsilon_i \in (0, \sigma_i)$ by the mean value theorem so that

$$\begin{aligned} &\frac{d}{d\epsilon}\left(\sum_F w(F) \sum_v \int_{\partial B_\epsilon(v)\cap F} |\nabla f|^2 ds\right)|_{\epsilon=\epsilon_i} \\ &= \frac{1}{\sigma_i} \sum_F w(F) \sum_v \int_{\partial B_{\sigma_i}\cap F} |\nabla f|^2 ds. \end{aligned}$$

Then

$$\begin{aligned} &\sum_F w(F) \sum_v \int_{\partial B_{\epsilon_i(v)}\cap F} \frac{\partial}{\partial r}|\nabla f|^2 ds \\ &= \sum_F w(F) \sum_v \epsilon_i \frac{d}{d\epsilon}\left(\frac{1}{\epsilon}\int_{\partial B_\epsilon(v)\cap F} |\nabla f|^2 ds\right)|_{\epsilon=\epsilon_i} \\ &= \frac{d}{d\epsilon}\left(\sum_F w(F) \sum_v \int_{\partial B_\epsilon(v)\cap F} |\nabla f|^2\right)|_{\epsilon=\epsilon_i} \\ &\quad - \sum_F w(F) \sum_v \frac{1}{\epsilon_i}\int_{\partial B_{\epsilon_i}(v)\cap F} |\nabla f|^2 ds \\ &= \sum_F w(F) \sum_v \frac{1}{\sigma_i}\int_{\partial B_{\sigma_i}(v)\cap F} |\nabla f|^2 \\ &\quad - \sum_F w(F) \sum_v \frac{1}{\epsilon_i}\int_{\partial B_{\epsilon_i}(v)\cap F} |\nabla f|^2 ds. \end{aligned}$$

Letting $i \to 0$, we obtain

$$0 \leq \int_X \triangle|\nabla f|^2 d\mu = \lim_{i\to 0} \sum_F w(F) \sum_v \int_{\partial B_{\epsilon_i(v)}\cap F} \frac{\partial}{\partial r}|\nabla f|^2 ds = L - L = 0.$$

q.e.d.

We are now ready to prove the following:

Theorem 8. *Suppose X is a 2-complex and (N, g) a complete Riemannian manifold of nonpositive sectional curvature. If $f : X \to (N, g)$ is a w-harmonic map and $|\nabla f|$ is bounded, then f is totally geodesic on each simplex of X. Furthermore, if the sectional curvature of N is strictly negative, then f maps each 2-simplex into a geodesic.*

Proof. By Theorem 7,

$$\int_X \triangle|\nabla f|^2 d\mu = 0.$$

On the other hand, for any point on the interior of a 2-simplex, the Bochner formula implies

$$\frac{1}{2}\triangle|\nabla f|^2 \geq |\nabla df|^2 \geq 0 \tag{10}$$

by the hypothesis on the flatness of X and the nonpositive curvature of N. Therefore $|\nabla df|^2 = 0$ and f is totally geodesic on each simplex. The second statement follows because the second inequality of (10) is a strict inequality if f is nonconstant and the sectional curvature of N is negative. q.e.d.

4. Tangent maps and eigenvalues of the link

By Theorem 8, a harmonic map is totally geodesic provided that the energy density function is bounded. This behavior can be guaranteed by certain assumption on the link of a vertex which is defined in terms of a spectral theory on graphs.

Let G be a graph. We denote the edges and vertices of G by $e_1, \ldots, e_L$ and $v_1, \ldots, v_K$ respectively. For each $k = 1, \ldots, K$, let $\mathcal{E}_k$ be the set of edges incident to vertex v_k. We identify each edge e_l with the interval $[0, \frac{\pi}{3}]$ (the $\frac{\pi}{3}$ corresponding to the fact that all our 2-simplices are equilateral triangles) and we assume that each edge e_l has an associated weight $\hat{w}_l = \hat{w}(e_l)$. In the case $G = \mathrm{Lk}(v)$ where v is a vertex of a 2-dimensional simplex X with weights $w(F_j)$, we define

$$\hat{w}(e_l) = w(F_l)$$

where F_l is the join (i.e., convex hull) of v and e_l. Returning to the case of a general graph G with weights $\hat{w}_l$, $l = 1, \ldots, L$, we let $\mathcal{G}$ be the set of functions $\varphi : G \to \mathbf{R}$ so that $\varphi_l = \varphi|_{e_l}$ is smooth up to the endpoints and

$$\sum_{e_l \in \mathcal{E}_k} \hat{w}_l \frac{\partial \varphi_l}{\partial \eta}(v_k) = 0 \tag{11}$$

where $\frac{\partial}{\partial \eta}$ is the outward pointing unit normal at the vertex v_k. On each edge e_l, define the measure $\hat{w}_l d\tau$, where $d\tau$ is the Lebesgue measure on $e_l \equiv [0, \frac{\pi}{3}]$, and let $d\nu$ be the measure on G so that $d\nu|_{e_l} = \hat{w}_l d\tau$.

Lemma 9. *For any $\varphi \in \mathcal{G}$,*

$$\int_G \psi\varphi'' d\nu = -\int_G \psi'\varphi' d\nu$$

where $\psi : G \to \mathbf{R}$ is a continuous function so that $\psi_l = \psi|_{e_l}$ is C^1.

Proof. By integration by parts, we get

$$\begin{aligned}\int_G \psi\varphi'' d\nu &= \sum_{l=1}^{L} \hat{w}_l \int_0^{\pi/3} \left((\psi_l\varphi_l')' - \psi_l'\varphi_l'\right) d\tau \\ &= \sum_{l=1}^{L} \hat{w}_l \left(\psi_l(0)\frac{d\varphi_l}{d\eta}(0) + \psi_l(\pi/3)\frac{d\varphi_l}{d\eta}(\pi/3) - \int_0^{\pi/3} \psi_l'\varphi_l' d\tau\right) \\ &= \sum_{l=1}^{L} \hat{w}_l \left(\psi_l(0)\frac{d\varphi_l}{d\eta}(0) + \psi_l(\pi/3)\frac{d\varphi_l}{d\eta}(\pi/3)\right) - \int_G \psi'\varphi' d\tau.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sum_{l=1}^{L} \hat{w}_l \left(\psi_l(0)\frac{d\varphi_l}{d\eta}(0) + \psi_l(\pi/3)\frac{d\varphi_l}{d\eta}(\pi/3)\right) \\ &= \sum_{k=1}^{K} \psi_l(v_k) \sum_{e_l \in \mathcal{E}_l} \hat{w}_l \frac{d\varphi_l}{d\eta}(v_k) = 0\end{aligned}$$

by equation (11). q.e.d.

Definition. The *eigenfunction* and *eigenvalue* (of the Laplacian $\triangle$ on G) is a function $\varphi \in \mathcal{G}$, not identically 0, and $\lambda \in \mathbf{R}$ so that for $\varphi_l = \varphi|_{e_l}$,

$$\varphi_l'' + \lambda\varphi_l = 0.$$

Lemma 10. *The eigenvalue λ of G is positive.*

Proof. By Lemma 9,

$$\lambda \int_G \varphi^2 d\nu = -\int_G \varphi\varphi'' d\nu = \int (\varphi')^2 d\nu,$$

which immediately implies $\lambda > 0$. q.e.d.

Next, we show that a tangent map of a w-harmonic map at 0-simplex v defines an eigenfunction on the link of v. First, we review the notion of tangent map f_* of f. For a more detailed discussion for the special case of w so that $w(F) = 1$, see [**DM**]. The general case considered here follows by a trivial modification of the argument in [**DM**]. Let $p \in X$ and d_g be the distance function on N induced by its metric g. Define

$$\begin{aligned}E(\sigma) &= \int_{B_\sigma(p)} |\nabla f|^2 d\mu, \\ I(\sigma) &= \int_{\partial B_\sigma(p)} d_g^2(f, f(p)) ds, \\ \mu(\sigma) &= (I(\sigma)\sigma^{-1})^{-\frac{1}{2}},\end{aligned}$$

and

$$\varrho = \lim_{\sigma \to 0} \frac{\sigma E(\sigma)}{I(\sigma)}$$

where ds is the measure on $\partial B_\sigma(p)$ so that $ds|_{F\cap\partial B_\sigma(p)} = w(F)d\theta$ where $d\theta$ is the Lebesgue measure on $\partial B_\sigma(p)$. The above limit always exists as the quotient appearing on the right is a monotone non-decreasing function of σ. This limit is called the *order* of f at p.

Let $p \in X$. The star $\mathrm{St}(p)$ is the union of all simplices whose closure contains p. Let σ be sufficiently small so that $B_\sigma(p) \subset \mathrm{St}(p)$, let B_1 be $B_\sigma(p)$ rescaled by a factor of $\frac{1}{\sigma}$ so that it has radius 1, and $S : B_1 \to B_\sigma(p)$ be the natural identification map defined by the scaling. Define the σ-blow up map of f at p as the map ${}^\sigma f : B_1 \to (N, g_\sigma)$ given by

$${}^\sigma f(p) = f \circ S(p)$$

and where (N, g_σ) is the manifold (N, g) rescaled by a factor of $\mu(\sigma)^2$, i.e., $g_\sigma = \mu(\sigma)^2 g$. The map ${}^\sigma f$ converges locally uniformly in the pull-back sense (see [**KS2**] and [**DM**]) to a w-harmonic homogeneous map $f_* = (f_*^1, \ldots, f_*^m) : B_1 \to \mathbf{R}^m$ of order ϱ. By using polar coordinates (r, θ) on each face F incident to p so that r measures the distance from the vertex p, we can write

$$f_*^\alpha(r, \theta) = r^\varrho f_*^\alpha(1, \theta)$$

for $\alpha = 1, \ldots, m$.

Lemma 11. *Let p be a vertex of X and $f_* : B_1 \to \mathbf{R}^n$ be a tangent map of f at p. The function $\varphi_* : G \to \mathbf{R}$ defined by letting $G = \partial B_1$ and $\varphi_*(\theta) = f_*^\alpha(1, \theta)$ is an eigenfunction of G with eigenvalue ϱ^2.*

Proof. Let v be a vertex in $\mathrm{Lk}(p)$. Let F_l, $l = 1, \ldots, L$, be the faces of B_1 which contain v. Without the loss of generality, we arrange the polar coordinates (r, θ) on F_l so that $(1, 0)$ corresponds to v. By Corollary 5,

$$\sum_{l=1}^{L} w(F_l) \frac{\partial f_{*l}^\alpha}{\partial \theta}(1, 0) = 0$$

where $f_{*l} = f_*|_{F_l}$, which is equivalent to

$$\sum_{e_l \in \mathcal{E}} \hat{w}_l \frac{\partial \varphi_{*l}^\alpha}{\partial \eta}(v) = 0$$

where $\mathcal{E}$ is the set of edges of $\mathrm{Lk}(p)$ containing v. Furthermore, f_*^α is a w-harmonic function on each face and hence

$$\begin{aligned} 0 &= \triangle f_*^\alpha \\ &= \frac{\partial^2 f_*^\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial f_*^\alpha}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f_*^\alpha}{\partial \theta^2} \\ &= \varrho(\varrho-1)r^{\varrho-2}\varphi_*^\alpha + \varrho r^{\varrho-2}\varphi_*^\alpha + r^{\varrho-2}(\varphi_*^\alpha)'' \\ &= (\varrho^2\varphi_*^\alpha + (\varphi_*^\alpha)'')r^{\varrho-2} \end{aligned}$$

which shows $(\varphi_*^\alpha)'' + \varrho^2 \varphi_*^\alpha = 0.$ q.e.d.

By combining Lemma 11 with Theorem 8, we obtain:

Theorem 12. *Suppose that X is a 2-complex such that every nonzero eigenvalue of the link of every vertex in X satisfies $\lambda \geq 1$.*

(i) *If $f : X \to (N, g)$ is a w-harmonic map into a complete Riemannian manifold of nonpositive sectional curvature, then f is totally geodesic on each 2-simplex of X. In particular, this implies that if the sectional curvature of N is negative, f maps each 2-simplex into a geodesic.*

(ii) *If the eigenvalues satisfy the stronger condition $\lambda > 1$ then f is a constant map.*

Proof. If ϱ is the order of f at a vertex p, then by Lemma 11, ϱ^2 is an eigenvalue. Hence $\varrho \geq 1$ by assumption and the first assertion of the theorem follows from Theorems 2 and 8. Let (r, θ) be the polar coordinate where r measures the distance from a vertex p. Since f is totally geodesic, we have $f(r, \theta) = rf(1, \theta)$. If f is not identically constant then f is of order 1 at p. Again, by Lemma 11, the second assertion of the theorem follows immediately. q.e.d.

Finally, we need to characterize the assumption on the complex X for which the eigenvalue assumption of Theorem 12 is satisfied. For this, we need to divert our attention to the notion of the discrete Laplacian on a graph. Again, let G be a weighted graph as in the beginning of this section. Let $V(G) = \{v_1, \dots, v_K\}$ denote the vertex set of G and let $A(G) \cong \mathbf{R}^K$ denote the space of functions $\varphi : V(G) \to \mathbf{R}$. Given $v_i, v_j \in G$, we set

$$\hat{w}_{ij} = \begin{cases} \hat{w}_s & \text{if } v_i \text{ is adjacent to } v_j \text{ and } v_i v_j = e_s \\ 0 & \text{otherwise} \end{cases}$$

and

$$d_i = \sum_{j=1}^{k} \hat{w}_{ij}.$$

Finally, the discrete Laplacian is defined to be the linear operator

$$\triangle^{\text{disc}} : A(G) \to A(G)$$

defined as

$$(\triangle^{\text{disc}} \varphi)(v_i) = \sum_{i,j=1}^{K} \frac{\hat{w}_{ij}}{d_i} (\varphi(v_i) - \varphi(v_j)).$$

Notice that $\triangle^{\text{disc}}$ is self-adjoint with respect to the inner product

$$\langle \varphi, \psi \rangle = \sum_{i=1}^{K} d_i \varphi(v_i) \psi(v_i)$$

and

$$\langle \triangle^{\mathrm{disc}} \varphi, \psi \rangle = \sum_{i=1}^{K} \sum_{i,j=1}^{K} \hat{w}_{ij} (\varphi(v_i) - \varphi(v_j))(\psi(v_i) - \psi(v_j)).$$

The next proposition relates the spectra of $\triangle$ and $\triangle^{\mathrm{disc}}$.

Proposition 13. *A real number* $\lambda \neq 9k^2$, $k = 1, 2, \ldots$, *is an eigenvalue of* $\triangle$ *if and only if* $1 - \cos\left(\frac{\sqrt{\lambda}\pi}{3}\right)$ *is an eigenvalue of* $\triangle^{disc}$.

Proof. Assume $\lambda > 0$ is an eigenvalue of $\triangle$ with eigenfunction φ, $\lambda \neq 9k^2$. Then for each $l = 1, \ldots, L$, $0 \leq x \leq \frac{\pi}{3}$,

$$\varphi_l(x) = \frac{\varphi_l(0) \sin(\sqrt{\lambda}(\pi/3 - x)) + \varphi_l(\pi/3) \sin \sqrt{\lambda} x}{\sin(\sqrt{\lambda}\pi/3)}.$$

We claim that the balancing condition implies that $\varphi|_{V(G)} : V(G) \to \mathbf{R}$ is an eigenfunction of $\triangle^{\mathrm{disc}}$ with eigenvalue $1 - \cos\sqrt{\lambda}$. Indeed, fix $v_i \in V(G)$. Then

$$\begin{aligned}
0 &= \sum_{e_l \in \mathcal{E}_i} w_l \frac{\partial \varphi_l}{\partial \eta}(0) \\
&= \sum_{e_l \in \mathcal{E}_i} w_l \frac{-\varphi_l(0)\sqrt{\lambda}\cos(\sqrt{\lambda}(\pi/3 - x)) + \varphi_l(\pi/3)\cos\sqrt{\lambda}x}{\sin(\sqrt{\lambda}\pi/3)}|_{x=0} \\
&= \frac{\sqrt{\lambda}}{\sin\left(\frac{\sqrt{\lambda}\pi}{3}\right)} \sum_{e_l \in \mathcal{E}_i} w_l \left(-\varphi_l(0)\cos\left(\frac{\sqrt{\lambda}\pi}{3}\right) + \varphi_l\left(\frac{\pi}{3}\right)\right) \\
&= \frac{\sqrt{\lambda}}{\sin\left(\frac{\sqrt{\lambda}\pi}{3}\right)} \left\{ -\cos\left(\frac{\sqrt{\lambda}\pi}{3}\right) \left(\sum_{e_l \in \mathcal{E}_i} w_l\right) \cdot \varphi(v_i) + \sum_{e_l \in \mathcal{E}_i} w_l \varphi_l\left(\frac{\pi}{3}\right) \right\} \\
&= \frac{\sqrt{\lambda}}{\sin\left(\frac{\sqrt{\lambda}\pi}{3}\right)} \left\{ -\cos\left(\frac{\sqrt{\lambda}\pi}{3}\right) d_i \varphi(v_i) + \sum_{j \neq i} \hat{w}_{ij} \varphi(v_j) \right\} \\
&= \frac{\sqrt{\lambda}}{\sin\left(\frac{\sqrt{\lambda}\pi}{3}\right)} \left\{ \left(1 - \cos\left(\frac{\sqrt{\lambda}\pi}{3}\right)\right) d_i \varphi(v_i) + \sum_{j} \hat{w}_{ij} (\varphi(v_j) - \varphi(v_i)) \right\},
\end{aligned}$$

hence

$$\sum_{i,j} \frac{\hat{w}_{ij}}{d_i} (\varphi(v_i) - \varphi(v_j)) = \left(1 - \cos\left(\frac{\sqrt{\lambda}\pi}{3}\right)\right) \varphi(v_i)$$

and the proof of the claim is complete. By reversing our argument the converse is also true. q.e.d.

Corollary 14. *Any nonzero eigenvalue of* $\triangle$ *is* $\geq (>) 1$ *if and only if the first nonzero eigenvalue of* $\triangle^{\mathrm{disc}}$ *is* $\geq (>) \frac{1}{2}$.

Proof. It follows immediately from Proposition 13 and the equivalence $1 - \cos\left(\frac{\pi}{3}\sqrt{\lambda}\right) \geq (>)\frac{1}{2}$ if and only if $\lambda \geq (>)1$ for $\lambda \geq 0$. q.e.d.

Now let Σ^n be a compact n-dimensional simplicial complex with an admissible weight c (see Definition 2.1 of [**W1**] for a precise definition). Let w be the induced weights on the 2-skeleton $\Sigma^{(2)} = X$. By applying Theorem 12 and Corollary 14 to X with weights w, we immediately obtain as a corollary the main theorem of [**W1**] (cf. [**W1**] Theorem 1.1)

Corollary 15. *Let (Σ^n, c) be a compact simplicial complex with admissible weight. Assume that the first nonzero eigenvalue of the link of vertex is $> \frac{1}{2}$. Then $\pi_1(\Sigma) = \Gamma$ has property F; i.e., any isometric action of Γ on a complete, simply connected manifold N of nonpositive sectional curvature has a fixed point on $\bar{N} = N \cup \partial N$.*

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