CORRELATION STRUCTURES, MANY-BODY SCATTERING PROCESSES AND THE DERIVATION OF THE GROSS-PITAEVSKII HIERARCHY

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ABSTRACT. We consider the dynamics of N bosons in three dimensions. We assume the pair interaction is given by $N^{3\beta-1}V(N^{\beta}\cdot)$. By studying an associated many-body wave operator, we introduce a BBGKY hierarchy which takes into account all of the interparticle singular correlation structures developed by the many-body evolution from the beginning. Assuming energy conditions on the N-body wave function, for $\beta \in (0, 1]$, we derive the Gross-Pitaevskii hierarchy with 2-body interaction. In particular, we establish that, in the $N \to \infty$ limit, all k-body scattering processes vanishes if $k \geqslant 3$ and thus provide a direct answer to a question raised by Erdös, Schlein, and Yau in [31]. Moreover, this new BBGKY hierarchy shares the limit points with the ordinary BBGKY hierarchy strongly for $\beta \in (0, 1)$ and weakly for $\beta = 1$. Since this new BBGKY hierarchy converts the problem from a two-body estimate to a weaker three-body estimate for which we have the estimates to achieve $\beta < 1$, it then allows us to prove that all limit points of the ordinary BBGKY hierarchy satisfy the space-time bound conjectured by Klainerman and Machedon in [47] for $\beta \in (0, 1)$.

CONTENTS

1. Introduction

A Bose-Einstein condensate (BEC), is a peculiar gaseous state in which particles of integer spin (bosons) occupy a macroscopic quantum state. Though the existence of a BEC was first predicted theoretically by Einstein for non-interacting particles in 1925, it was not verified experimentally until the Nobel prize winning first observation of Bose-Einstein condensate (BEC) for interacting atoms in low temperature in 1995 [4, 26] using laser cooling techniques. Since then, this new state of matter has attracted a lot of attention in physics and mathematics as it can be used to explore fundamental questions in quantum mechanics, such as the emergence of interference, decoherence, superfluidity and quantized vortices. Investigating various condensates has become one of the most active areas of contemporary research.

As in the study of any time-dependent interacting N-body system, the main difficulty in the theory of BEC is that the governing PDE is impossible to solve or simulate when N is large. For BEC, the time-evolution of a N boson system without trapping in $\mathbb{R}^3$ is governed by the many-body Schrödinger equation

$$i\partial_t \psi_N = H_N \psi_N \tag{1.1}$$

where the N-body Hamiltonian is given by

$$H_N = -\sum_{j=1}^{N} \triangle_{x_j} + \sum_{1 \leqslant i < j \leqslant N} N^{3\beta-1} V(N^\beta (x_i - x_j)) \text{ with } \beta \in (0,1] . \tag{1.2}$$

Here, $(x_1, ..., x_N) \in \mathbb{R}^{3N}$ is the position vector of N particles in $\mathbb{R}^3$, we choose $\|\psi_N(0)\|_{L^2(\mathbb{R}^{3N})} = 1$, and we assume the interparticle interaction is given by $N^{3\beta-1} V(N^\beta \cdot)$. On the one hand,

$$V_N(\cdot) = N^{3\beta} V(N^\beta \cdot) \tag{1.3}$$

is an approximation of the Dirac δ-function as $N \to \infty$ and hence matches the Gross-Pitaevskii description that the many-body effect should be modeled by an on-site strong self interaction.[1] On the other hand, if we denote by $\mathrm{scat}(W)$ the 3D scattering length of the potential W, then we have

$$N \, \mathrm{scat}(N^{-1} V_N(\cdot)) \sim 1$$

which is the Gross-Pitaevskii scaling condition introduced by Lieb, Seiringer and Yngvason in [50]. In the current experiments, we have $N \sim 10^4$ which already makes equation (1.1) unrealistic to solve. In fact, according to the references in [50], the largest system one could simulate at the moment has $N \sim 10^2$. Hence, it is necessary to find reductions or approximations.

It is widely believed that the mean-field approximation / limit of equation (1.1) is given by the cubic nonlinear Schrödinger equation (NLS)

$$i\partial_t \phi = -\triangle \phi + c \left| \phi \right|^2 \phi, \tag{1.4}$$

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[1]From here on out, we consider the $\beta > 0$ case solely. For $\beta = 0$ (Hartree dynamics), see [34, 29, 48, 53, 51, 39, 40, 17, 2, 3, 8] .

where the coupling constant c is exactly given by $8\pi N \operatorname{scat}(N^{-1}V_N(\cdot))$. That is, if we define the k-particle marginal densities associated with ψ_N by

$$(1.5) \qquad \gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') = \int \psi_N(t, \mathbf{x}_k, \mathbf{x}_{N-k})\overline{\psi_N}(t, \mathbf{x}_k', \mathbf{x}_{N-k})d\mathbf{x}_{N-k}, \quad \mathbf{x}_k, \mathbf{x}_k' \in \mathbb{R}^{3k},$$

and assume

$$\gamma_N^{(k)}(0, \mathbf{x}_k, \mathbf{x}_k') \sim \prod_{j=1}^{k} \phi_0(x_j)\bar{\phi}_0(x_j') \text{ as } N \to \infty$$

where $\mathbf{x}_k = (x_1, ..., x_j) \in \mathbb{R}^{3k}$, then we have the propagation of chaos, namely,

$$(1.6) \qquad \gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k') \sim \prod_{j=1}^{k} \phi(t, x_j)\bar{\phi}(t, x_j') \text{ as } N \to \infty$$

and $\phi(t, x_j)$ is given by (1.4) subject to the initial $\phi(0, x_j) = \phi_0(x_j)$. Naturally, to prove (1.6), one studies the $N \to \infty$ limit of the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy of the many-body system (1.1) satisfied by $\left\{\gamma_N^{(k)}\right\}$:

$$(1.7) \qquad i\partial_t \gamma_N^{(k)} + \left[\triangle_{\mathbf{x}_k}, \gamma_N^{(k)}\right] = \frac{1}{N} \sum_{1 \leqslant i < j \leqslant k} \left[V_N(x_i - x_j), \gamma_N^{(k)}\right]$$
$$+ \frac{N-k}{N} \sum_{j=1}^{k} \operatorname{Tr}_{k+1} \left[V_N(x_j - x_{k+1}), \gamma_N^{(k+1)}\right]$$

if we do not distinguish $\gamma_N^{(k)}$ as a kernel and the operator it defines. Here the operator $V_N(x)$ represents multiplication by the function $V_N(x)$ and Tr_{k+1} means taking the $k+1$ trace, for example,

$$\operatorname{Tr}_{k+1} V_N(x_j - x_{k+1})\gamma_N^{(k+1)} = \int V_N(x_j - x_{k+1})\gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}_k', x_{k+1})dx_{k+1}.$$

Such an approach for deriving mean-field type equations by studying the limit of the BBGKY hierarchy was proposed by Kac in the classical setting and demonstrated by Landford's work on the Boltzmann equation. In the current quantum setting, it was suggested by Spohn [54] and has been proven to be successful by Erdös, Schlein, and Yau in their fundamental papers [30, 31, 32, 33] which have inspired many works by many authors [47, 45, 11, 18, 13, 19, 7, 20, 21, 38, 22, 56, 23] .

This paper, like the aforementioned work, is inspired by the work of Erdös, Schlein, and Yau. The first main part of this paper deals with a problem raised on [31, p.516]. To motivate and state the problem, we first notice the formal limit of hierarchy (1.7):

$$(1.8) \qquad i\partial_t \gamma^{(k)} + \left[\triangle_{\mathbf{x}_k}, \gamma^{(k)}\right] = b_0 \sum_{j=1}^{k} \operatorname{Tr}_{k+1} \left[\delta(x_j - x_{k+1}), \gamma^{(k+1)}\right]$$

where

$$b_0 = \int_{\mathbb{R}^3} V(x)dx.$$

We make such an observation because $V_N(\cdot) \to \left(\int_{\mathbb{R}^3} V(x) dx \right) \delta(\cdot)$. If we plug

$$(1.9) \qquad\qquad \gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) = \prod_{j=1}^{k} \phi(t, x_j) \bar{\phi}(t, x'_j)$$

into (1.8) and assume ϕ solves (1.4), then (1.9) is a solution to (1.8) if and only if the coupling constant c in (1.4) equals to b_0. Since

$$8\pi \lim_{N \to \infty} N \operatorname{scat}(N^{-1} V_N(\cdot)) = b_0 \text{ for } \beta \in (0, 1),$$

the formal limit (1.8) checks the prediction. It also has been proven in [31] for $\beta \in (0, 1/2)$. However, this formal limit does not meet the prediction when $\beta = 1$ because

$$8\pi N \operatorname{scat}(N^{-1} V_N(\cdot)) = 8\pi \operatorname{scat}(V) \equiv 8\pi a_0 \text{ for } \beta = 1$$

which is usually a number smaller than b_0. In [30, 32, 33], Erdös, Schlein and Yau have established rigorously that the real limit of the BBGKY hierarchy (1.7) associated with (1.1) matches the prediction and is given by

$$(1.10) \qquad i\partial_t \gamma^{(k)} + \left[\triangle_{\mathbf{x}_k}, \gamma^{(k)} \right] = 8\pi a_0 \sum_{j=1}^{k} \operatorname{Tr}_{k+1} \left[\delta(x_j - x_{k+1}), \gamma^{(k+1)} \right].$$

The reasoning given is that one has to take into account the correlation between the particles. To be specific, as in [50, 30, 31, 33], let w_0 be the solution to

$$(-\triangle + \frac{1}{2} N^{\beta-1} V) w_0(x) = \frac{1}{2} V,$$
$$\lim_{|x| \to \infty} w_0(x) = 0.$$

We scale w_0 by

$$w_N(x) = N^{\beta-1} w_0 \left(N^{\beta} x \right)$$

so that w_N is the solution to

$$(1.11) \qquad\qquad (-\triangle + \tfrac{1}{2N} V_N)(1 - w_N(x)) = 0,$$
$$\lim_{|x| \to \infty} w_N(x) = 0.$$

The papers [30, 32, 33] then suggest that, instead of considering the limit of hierarchy (1.7) directly, one should investigate the limit of the following hierarchy

$$(1.12) \qquad i\partial_t \gamma_N^{(k)} + \triangle_{\mathbf{x}_k} \gamma_N^{(k)} - \triangle_{\mathbf{x}'_k} \gamma_N^{(k)}$$
$$= \frac{1}{N} \sum_{1 \leqslant i < j \leqslant k} \left(\tilde{V}_N(x_i - x_j) \gamma_{N,i,j}^{(k)} - \tilde{V}_N(x'_i - x'_j) \gamma_{N,i',j'}^{(k)} \right)$$
$$+ \frac{N-k}{N} \sum_{j=1}^{k} \operatorname{Tr}_{k+1} \left(\tilde{V}_N(x_j - x_{k+1}) \gamma_{N,j,k+1}^{(k+1)} - \tilde{V}_N(x'_j - x'_{k+1}) \gamma_{N,j',(k+1)'}^{(k+1)} \right),$$

which has the singular correlations between particles built in. Here

$$\tilde{V}_N(\cdot) = V_N(\cdot)(1 - w_N(\cdot)),$$

and

$$\gamma_{N,i,j}^{(k)} = \frac{\gamma_N^{(k)}}{(1 - w_N(x_i - x_j))}.$$

As $N \to \infty$, one formally has

$$\gamma_N^{(k)} \sim \gamma_{N,i,j}^{(k)},$$

and

$$\tilde{V}_N(\cdot) \to 8\pi a_0 \delta(\cdot),$$

hence one obtains (1.10) as the limit of the many-body dynamic (1.1).

One immediate question to this delicate limiting process is: aside from physical motivation, is there a more mathematical explanation for why (1.8) is not the limit of (1.1) when $\beta = 1$? An answer is that the "usual" energy condition:

(1.13) $$\sup_t \left(\operatorname{Tr} S^{(k+1)} \gamma_N^{(k+1)} + \frac{1}{N} \operatorname{Tr} S_1 S_{1'} S^{(k)} \gamma_N^{(k)} \right) \leqslant C^k \text{ for } k \geqslant 0,$$

where

$$S_j = (1 - \triangle_{x_j})^{\frac{1}{2}} \text{ and } S^{(k)} = \prod_{j=1}^k S_j S_{j'},$$

first proved in [28, 31] for $\beta \in \left(0, \frac{3}{5}\right)$ and later in [45, 11, 18, 19, 22, 23] , is not true when $\beta = 1$. This can be proved by contradiction: assume that (1.13) does hold when $\beta = 1$, then with a simple argument in [45] which is first hinted in [31] and used in [45, 11, 18, 19, 22, 23] , one easily proves that hierarchy (1.7) converges to the wrong limit (1.8) and reaches a contradiction.

Another immediate but much deeper question is that, if the singular correlation structure between particles is so crucial, then why would one only take a pair into account at a time? For example, when considering the term

$$V_N(x_1 - x_2)\gamma_N^{(k)}$$

why would one only put in the singular correlation structure between particles x_1 and x_2 and why not put in the singular correlation structure between particles x_1 and x_3 or x_2 and x_3? That is, why not consider a term like

$$\left[\tilde{V}_N(x_1 - x_2)(1 - w_N(x_1 - x_3))\right] \left[\frac{\gamma_{N,1,2}^{(k)}}{(1 - w_N(x_1 - x_3))}\right]?$$

The above expression corresponds to a three-body interaction. Basically, the question is: why can this case be dropped? This is actually a problem raised on [31, p.516].

Problem 1 ([31, p.516]). *One should rigorously establish the fact that all three-body scattering processes are negligible in the limit.*

In the first main part of this paper, we provide a direct answer to Problem 1. We take into account all of the interparticle singular correlation structures developed by the many-body

evolution from the beginning.[2] We rigorously establish that, in the $N \to \infty$ limit, all k-body scattering processes vanishes if $k \geqslant 3$. To be specific, we have the following theorem.

Theorem 1.1 (Main Theorem I). *Define*

$$\text{(1.14)} \qquad \alpha_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k') \stackrel{\text{def}}{=} \left(G_N^{(k)}(\mathbf{x}_k) \right)^{-1} \left(G_N^{(k)}(\mathbf{x}_k') \right)^{-1} \gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k'),$$

where

$$\text{(1.15)} \qquad G_N^{(k)}(\mathbf{x}_k) \stackrel{\text{def}}{=} \prod_{1 \leq i < j \leq k} (1 - w_N(x_i - x_j)).$$

Suppose $\beta \in (0,1]$. Assume the energy bound[3]:

$$\text{(1.16)} \qquad \sup_t \left(\text{Tr } S^{(3)} \alpha_N^{(3)} + \frac{1}{N} \text{Tr } S_1 S_{1'} S^{(3)} \alpha_N^{(3)} \right) \leqslant C.$$

Moreover, denote $\mathcal{L}_k^2$ the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^{3k})$. Then every limit point $\Gamma = \{\gamma^{(k)}\}_{k=1}^{\infty}$ of $\left\{ \Gamma_N(t) = \{\alpha_N^{(k)}\}_{k=1}^{N} \right\}$ in $\bigoplus_{k \geqslant 1} C\left([0,T], \mathcal{L}_k^2\right)$ with respect to the product topology τ_{prod} (defined in Appendix A), if there is any, satisfies the cubic Gross-Pitaevskii hierarchy:

$$\text{(1.17)} \qquad i\partial_t \gamma^{(k)} = \sum_{j=1}^{k} \left[-\triangle_{x_j}, \gamma^{(k)} \right] + c_0 \sum_{j=1}^{k} \text{Tr}_{k+1} \left[\delta(x_j - x_{k+1}), \gamma^{(k+1)} \right],$$

where the coupling constant c_0 is given by

$$\text{(1.18)} \qquad c_0 = \begin{cases} \int_{\mathbb{R}^3} V(x) dx & \text{if } \beta \in (0,1), \\ 8\pi a_0 & \text{if } \beta = 1. \end{cases}$$

An important feature of $\alpha_N^{(k)}$ is that, considered as bounded operators, $\alpha_N^{(k)}$ and $\gamma_N^{(k)}$ share the same $N \to \infty$ limit for $\beta \in (0,1)$, if there is any.[4] We will prove this simple fact in Lemma 2.1, §2. Hence, Theorem 1.1 and its proof give us a better understanding of the limiting process and allow us to solve an open problem, raised by Klainerman and Machedon in 2008, for $\beta \in (0,1)$ in the second main part of this paper. After reading Theorem 1.1, an alert reader can easily tell that one needs to prove a uniqueness theorem of solutions to hierarchy (1.17) before concluding that equation (1.4) is the mean-field limit to the N-body dynamic (1.1). In the second main part of this paper, we solve an open problem about an a-priori bound on the limit points which leads to uniqueness of (1.17), conjectured by Klainerman and Machedon [47] in 2008 for $\beta \in (0,1)$. Though this conjecture was not stated explicitly in [47], as we will explain after stating Theorem 1.2, this Klainerman-Machedon a-priori

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[2]In the Fock space version of the problem, there is another way to insert all of the correlation structures using the metaplectic representation / Bogoliubov transform. See [7].

[3]We remind the readers that the "usual" energy condition (1.13) is not true when $\beta = 1$. The energy conditions (1.16) and (1.19) we impose on Theorems 1.1 and 1.2 have been proven for $k = 0, 1$ or with spatial cut-offs for general k in [33, 32].

[4]The same thing is weakly true for $\beta = 1$ but we omit the proof at the moment since Theorem 1.2 applies only to $\beta < 1$.

bound is necessary to implement Klainerman-Machedon's powerful and flexible approach in the most involved part of proving the cubic nonlinear Schrödinger equation (NLS) as the $N \to \infty$ limit of quantum N-body dynamics. Kirkpatrick-Schlein-Staffilani [45] completely solved the $\mathbb{T}^2$ version of the conjecture with a trace theorem and were the first to successfully implement such an approach. However, the $\mathbb{R}^3$ version of the conjecture as stated inside Theorem 1.2, was fully open until recently. T. Chen and Pavlović [13] have been able to prove the conjecture for $\beta \in (0, 1/4)$. In [19], X.C simplified and extended the result to the range of $\beta \in (0, 2/7)$. X.C. and J.H. [21] then extended the $\beta \in (0, 2/7)$ result by X.C. to $\beta \in (0, 2/3)$. In the second main part of this paper, we prove it for $\beta \in (0, 1)$. In particular, away from the $\beta = 1$ case, the conjecture is now resolved. To be specific, we prove the following theorem.

Theorem 1.2 (Main Theorem II). *Define*

$$R^{(k)} = \prod_{j=1}^{k} \left|\nabla_{x_j}\right| \left|\nabla_{x_j'}\right|,$$

and

$$B_{j,k+1}\gamma^{(k+1)} = \mathrm{Tr}_{k+1}\left[\delta(x_j - x_{k+1}), \gamma^{(k+1)}\right],$$

Suppose $\beta \in (0,1)$. Assume the energy bound:

$$(1.19) \qquad \sup_t \left(\mathrm{Tr}\, S^{(k+1)}\alpha_N^{(k+1)} + \frac{1}{N}\mathrm{Tr}\, S_1 S_{1'} S^{(k)}\alpha_N^{(k)}\right) \leqslant C_0^{k+1} \text{ for } k \geqslant 0,$$

then every limit point $\Gamma = \left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ of $\left\{\Gamma_N(t) = \left\{\alpha_N^{(k)}\right\}_{k=1}^{N}\right\}$ obtained in Theorem 1.1 (and hence of $\left\{\left\{\gamma_N^{(k)}\right\}_{k=1}^{N}\right\}_{N=1}^{\infty}$ because they have the same limit), satisfies the space-time bound conjectured by Klainerman-Machedon [47] in 2008:

$$(1.20) \qquad \int_0^T \left\|R^{(k)} B_{j,k+1}\gamma^{(k+1)}(t,\cdot,\cdot)\right\|_{L^2_{\mathbf{x},\mathbf{x}'}} dt \leqslant C^k.$$

In particular, there is only one limit point due to the Klainerman-Machedon uniqueness theorem [47, Theorem 1.1].

In 2007, Erdös, Schlein, and Yau obtained the first uniqueness theorem of solutions [31, Theorem 9.1] to hierarchy (1.17). The proof is surprisingly delicate – it spans 63 pages and uses complicated Feynman diagram techniques. The main difficulty is that hierarchy (1.17) is a system of infinitely coupled equations. Briefly, [31, Theorem 9.1] is the following:

Theorem 1.3 (Erdös-Schlein-Yau uniqueness [31, Theorem 9.1]). *There is at most one nonnegative symmetric operator sequence $\left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ that solves hierarchy (1.17) subject to the energy condition*

$$(1.21) \qquad \sup_{t \in [0,T]} \mathrm{Tr}\, S^{(k)}\gamma^{(k)} \leqslant C^k.$$

In [47], based on their null form paper [46], Klainerman and Machedon gave a different uniqueness theorem of hierarchy (1.17) in a space different from that used in [31, Theorem 9.1]. The proof is shorter (13 pages) than the proof of [31, Theorem 9.1]. Briefly, [47, Theorem 1.1] is the following:

Theorem 1.4 (Klainerman-Machedon uniqueness [47, Theorem 1.1]). *There is at most one symmetric operator sequence $\left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ that solves hierarchy (1.17) subject to the space-time bound (1.20).*

When propagation of chaos (1.6) happens, condition (1.21) is actually

$$(1.22) \qquad\qquad \sup_{t\in[0,T]} \|\langle \nabla_x \rangle \phi\|_{L^2} \leqslant C,$$

while condition (1.20) means

$$(1.23) \qquad\qquad \int_0^T \left\||\nabla_x| \left(|\phi|^2 \phi\right)\right\|_{L^2} dt \leqslant C.$$

When ϕ satisfies NLS (1.4), both are known. Due to the Strichartz estimate [43], (1.22) implies (1.23), that is, condition (1.20) seems to be a bit weaker than condition (1.21). The proof of [47, Theorem 1.1] (13 pages) is also considerably shorter than the proof of [31, Theorem 9.1] (63 pages). It is then natural to wonder whether [47, Theorem 1.1] provides a simple proof of uniqueness. To answer such a question it is necessary to know whether the limit points in Theorem 1.1 satisfy condition (1.20).

Away from curiosity, there are realistic reasons to study the Klainerman-Machedon bound (1.20). In the NLS literature, uniqueness subject to condition (1.22) is called unconditional uniqueness while uniqueness subject to condition (1.23) is called conditional uniqueness. While the conditional uniqueness theorems usually come for free with the uniqueness conditions verified naturally in NLS theory because they are parts of the existence argument, the unconditional uniqueness theorems usually do not yield any information of existence. Recently, using a version of the quantum de Finetti theorem from [49], T. Chen, Hainzl, Pavlović, and Seiringer [15] provided an alternative 33 pages proof to [31, Theorem 9.1] and confirmed that it is an unconditional uniqueness result in the sense of NLS theory.[5] Therefore, the general existence theory of the Gross-Pitaevskii hierarchy (1.17) subject to general initial datum has to require that the limits of the BBGKY hierarchy (1.7) lie in the space in which the space-time bound (1.20) holds. See [10, 12, 13, 14] .

Moreover, while [31, Theorem 9.1] is a powerful theorem, it is very difficult to adapt such an argument to various other interesting and colorful settings: a different spatial dimension, a three-body interaction instead of a pair interaction, or the Hermite operator instead of the Laplacian. The last situation mentioned is physically important. On the one hand, all the known experiments of BEC use harmonic trapping to stabilize the condensate [4, 26, 9, 44, 55]. On the other hand, different trapping strength produces quantum behaviors which do not exist in the Boltzmann limit of classical particles nor in the quantum case when the trapping is missing and have been experimentally observed [35, 57, 25, 41, 27]. The Klainerman-Machedon

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[5]See also [56, 24, 42].

approach applies easily in these meaningful situations ([45, 11, 18, 19, 20, 36, 22, 23]). Thus proving the Klainerman-Machedon bound (1.20) actually helps to advance the study of quantum many-body dynamic and the mean-field approximation in the sense that it provides a flexible and powerful tool in 3D.

1.1. **Organization of the Paper.** We will first compute the BBGKY hierarchy satisfied by $\left\{\alpha_N^{(k)}\right\}$, defined in (1.14), in §2. Due to the definition of $\left\{\alpha_N^{(k)}\right\}$, the BBGKY hierarchy of $\left\{\alpha_N^{(k)}\right\}$, written as (2.13), takes into account all of the singular correlation structures developed by the many-body evolution from the beginning. The differences between hierarchy (2.13) and hierarchy (1.7) are obvious: hierarchy (2.13) for $\alpha_N^{(k)}$ has $k-$body interactions where $k = 2, ..., k$, but most importantly, for the purpose of Theorems 1.1 and 1.2, hierarchy (2.13) does not have 2-body interactions not under an integral sign. We will call the key new terms the *potential terms*, which consist of three-body interactions, and the *k-body interaction terms*, which consist of k-body interaction for all $k \geqslant 3$.

With the BBGKY hierarchy satisfied by $\left\{\alpha_N^{(k)}\right\}$ computed in §2, we prove Theorem 1.1 in §3 as a "warm up" first and then establish Theorem 1.2 in §4. The gut of the proof of Theorems 1.1 and 1.2 is the careful application of the 3D and 6D retarded endpoint Strichartz estimates [43] and the Littlewood-Paley theory.

One of the effects of considering the singular interparticle correlation structures developed by the many-body evolution is to replace the potential

$$(1.24) \qquad N^{-1}V_N(x_i - x_j) = N^{3\beta-1}V(N^\beta(x_i - x_j))$$

with the new potential

$$(1.25) \qquad (\nabla_{x_\ell} G_{N,i,\ell})(\nabla_{x_\ell} G_{N,j,\ell}), \quad i \neq j, i \neq \ell, j \neq \ell$$

(among other terms). (1.25) could be considered as a three-body interaction, since it is only nontrivial if all three x_i, x_j, and x_ℓ are within $\sim N^{-\beta}$. One might wonder why a three-body interaction is better then a two-body interaction because a three-body interaction is more complicated. For the purposes of estimates, the original potential (1.24) has the behavior

$$(1.26) \qquad N^{-1}V_N(x_i - x_j) \sim N^{3\beta-1}\langle N^\beta(x_i - x_j)\rangle^{-100}$$

For the new potential, we have effectively

$$(1.27) \qquad (\nabla_{x_\ell} G_{N,i,\ell})(\nabla_{x_\ell} G_{N,j,\ell}) \sim N^{4\beta-2}\langle N^\beta(x_i - x_\ell)\rangle^{-2}\langle N^\beta(x_j - x_\ell)\rangle^{-2}$$

Note that if $\beta = 1$ and $i = j$, then (1.27) and (1.26) are effectively the same, and there is no gain in going from (1.24) to (1.25). However, $i \neq j$ in (1.25) and hence (1.25), a three-body interaction, actually offers more localization than (1.24), a two-body interaction. It is then natural to use the 6D endpoint Strichartz estimate when one wants to estimate a term like

$$\left\| \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \left[(\nabla_{x_\ell} G_{N,i,\ell})(\nabla_{x_\ell} G_{N,j,\ell}) \alpha_N^{(k)}(t_{k+1}) \right] dt_{k+1} \right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}.$$

Here $U^{(k)}(t_k) = e^{it_k \triangle_{\mathbf{x}_j}} e^{-it_k \triangle_{\mathbf{x}'_j}}$.

Using the Littlewood-Paley theory or frequency localization effectively gains one derivative in the analysis. That is, we avoid a N^β in the estimates. Heuristically speaking, it sort of averages the best and the worst estimates. Here, the "best" means no derivatives hits V_N and the "worst" means that two derivatives hit V_N. For example, say one would like to look at

$$(1.28) \qquad \left\| P_M^1 P_M^2 \left| \nabla_{x_1} \right| \left| \nabla_{x_2} \right| V_N(x_1 - x_2) \right\|_{L^2}.$$

Here, P_M^i is the Littlewood-Paley projection onto frequencies $\sim M$, acting on functions of $x_i \in \mathbb{R}^3$. There are two ways to look at (1.28), namely

$$M^2 \left\| V_N(x_1 - x_2) \right\|_{L^2}$$

and

$$N^{2\beta} \left\| P_M^1 P_M^2 \left(V'' \right)_N (x_1 - x_2) \right\|_{L^2}.$$

Then depending on the sizes of N^β and M, one is better than the other. As we will see in the proof of Theorem 1.2 in §4, such a consideration will effectively avoid a N^β in the estimates.

2. The BBGKY Hierarchy with Singular Correlation Structure

Recall (1.15)

$$G_N^{(k)}(x_1, \cdots, x_N) = \prod_{1 \leq i < j \leq k} (1 - w_N(x_i - x_j)),$$

where w_N is defined via (1.11). We decompose $G_N^{(k)}$ as follows:

$$G_N^{(k)} = \prod_{1 \leq i < j \leq k} G_{N,i,j}, \qquad G_{N,i,j} = 1 - w_N(x_i - x_j)$$

and define the multiplication operator $Y_N^{(k)}$ by

$$(2.1) \qquad ((Y_N^{(k)})^{-1} \psi_N)(x_1, \cdots, x_N) \overset{\text{def}}{=} G_N^{(k)}(x_1, \cdots, x_k) \psi_N(x_1, \cdots, x_N).$$

An immediate property of $Y_N^{(k)}$ is the following.

Lemma 2.1. *Let $\alpha_N^{(k)}$ be defined as in (1.14). For $\beta \in (0, 1)$, $\forall f \in L^2\left(\mathbb{R}^{3k}\right)$,*

$$\lim_{N \to \infty} \left\| \alpha_N^{(k)}(f) - \gamma_N^{(k)}(f) \right\|_{L^2} \leq C \|f\| \lim_{N \to \infty} \left\| Y_N^{(k)} - 1 \right\|_{op} = 0.$$

Here $\alpha_N^{(k)}(f)$ and $\gamma_N^{(k)}(f)$ means the operators $\alpha_N^{(k)}$ and $\gamma_N^{(k)}$ act on f, and $\|\cdot\|_{op}$ means the operator norm.

Proof. We have

$$\begin{aligned}
\left\| \alpha_N^{(k)} f - \gamma_N^{(k)} f \right\|_{L^2} &= \left\| Y_N^{(k)} \gamma_N^{(k)} Y_N^{(k)} f - \gamma_N^{(k)} f \right\|_{L^2} \\
&\leq \left\| Y_N^{(k)} \gamma_N^{(k)} Y_N^{(k)} f - \gamma_N^{(k)} Y_N^{(k)} f \right\|_{L^2} + \left\| \gamma_N^{(k)} Y_N^{(k)} f - \gamma_N^{(k)} f \right\|_{L^2} \\
&\leq \left\| Y_N^{(k)} - 1 \right\|_{op} \left\| \gamma_N^{(k)} \right\|_{op} \left\| Y_N^{(k)} \right\|_{op} \|f\|_{L^2} + \left\| \gamma_N^{(k)} \right\|_{op} \left\| Y_N^{(k)} - 1 \right\|_{op} \|f\|_{L^2}.
\end{aligned}$$

Notice that the Hilbert-Schmidt norm of $\gamma_N^{(k)}$ is uniformly bounded by 1 because we assume $\|\psi_N(0)\|_{L^2(\mathbb{R}^{3N})} = 1$. Moreover, since $\beta < 1$, we have

$$\lim_{N\to\infty} \left\| Y_N^{(k)} - 1 \right\|_{op} = 0.$$

In fact, consider

$$\int \left| \frac{f(x_1, x_2)}{1 - \omega_N(x_1 - x_2)} - f(x_1, x_2) \right|^2 d\mathbf{x}_2 = \int \left| \frac{\omega_N(x_1 - x_2)}{1 - \omega_N(x_1 - x_2)} \right|^2 |f(x_1, x_2)|^2 \, d\mathbf{x}_2$$

where

$$\frac{\omega_N(x_1 - x_2)}{1 - \omega_N(x_1 - x_2)} = \frac{N^{\beta-1}\omega_0(N^\beta(x_1 - x_2))}{1 - N^{\beta-1}\omega_0(N^\beta(x_1 - x_2))}$$

$$\leqslant \ CN^{\beta-1} \to 0 \text{ as } N \to \infty$$

because $\omega_0 \in L^\infty(\mathbb{R}^3)$ and $\beta < 1$. So we conclude that

$$\lim_{N\to\infty} \left\| \alpha_N^{(k)}(f) - \gamma_N^{(k)}(f) \right\|_{L^2} \leqslant C \|f\| \lim_{N\to\infty} \left\| Y_N^{(k)} - 1 \right\|_{op} = 0.$$

$\square$

To compute the BBGKY hierarchy of $\left\{ \alpha_N^{(k)} \right\}$, we first give the following lemma.

Lemma 2.2. *We have*

(2.2) $$H_N^{(k)}(Y_N^{(k)})^{-1} = (Y_N^{(k)})^{-1}(H_{N,0}^{(k)} + A_N^{(k)} + E_N^{(k)}),$$

where $H_{N,0}^{(k)}$ is the ordinary Laplacian

$$H_{N,0}^{(k)} = -\sum_{j=1}^{k} \Delta_{x_j},$$

$A_N^{(k)}$ *is the zeroth order operator of multiplication by*

$$-\sum_{\substack{1 \leq i,j,\ell \leq k \\ i,j,\ell \text{ distinct}}} \frac{\nabla_{x_\ell} G_{N,\ell,i} \cdot \nabla_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,i} \, G_{N,\ell,j}},$$

and $E_N^{(k)}$ is the first order operator

$$2\sum_{\substack{1 \leq j,\ell \leq k \\ j \neq \ell}} \frac{\nabla_{x_\ell} G_{N,j,\ell}}{G_{N,j,\ell}} \cdot \nabla_{x_\ell}.$$

Before proceeding to the proof, let us note that the terms $A_N^{(k)}$ and $E_N^{(k)}$ should be thought of as "error terms". Indeed, $A_N^{(k)}$ involves only three-body interaction – it is only nontrivial if x_i, x_j, and x_ℓ are within $\sim N^{-\beta}$ of each other.

Proof. We start with

$$(2.3) \qquad (-\Delta_{x_\ell}) \log G_N^{(k)} = -\frac{\Delta_{x_\ell} G_N^{(k)}}{G_N^{(k)}} + \frac{|\nabla_{x_\ell} G_N^{(k)}|^2}{(G_N^{(k)})^2}$$

Using that

$$\frac{\nabla_{x_\ell} G_N^{(k)}}{G_N^{(k)}} = \sum_{1 \le i < j \le k} \frac{\nabla_{x_\ell} G_{N,i,j}}{G_{N,i,j}} = \sum_{\substack{1 \le j \le k \\ j \neq \ell}} \frac{\nabla_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,j}},$$

we can rewrite (2.3) as

$$(2.4) \qquad -\frac{\Delta_{x_\ell} G_N^{(k)}}{G_N^{(k)}} = (-\Delta_{x_\ell}) \log G_N^{(k)} - \left| \sum_{\substack{1 \le j \le k \\ j \neq \ell}} \frac{\nabla_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,j}} \right|^2.$$

On the other hand, we have

$$\log G_N^{(k)} = \sum_{1 \le i < j \le k} \log G_{N,i,j},$$

and hence (2.3) also reads

$$\begin{aligned}
(-\Delta_{x_\ell}) \log G_N^{(k)} &= \sum_{1 \le i < j \le k} \left(-\frac{\Delta_{x_\ell} G_{N,i,j}}{G_{N,i,j}} + \left| \frac{\nabla_{x_\ell} G_{N,i,j}}{G_{N,i,j}} \right|^2 \right) \\
&= \sum_{\substack{1 \le j \le k \\ j \neq \ell}} \frac{-\Delta_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,j}} + \sum_{\substack{1 \le j \le k \\ j \neq \ell}} \left| \frac{\nabla_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,j}} \right|^2.
\end{aligned}$$

Plugging this into (2.4) and expanding the square in (2.4),

$$-\frac{\Delta_{x_\ell} G_N^{(k)}}{G_N^{(k)}} = \sum_{\substack{1 \le j \le k \\ j \neq \ell}} \frac{-\Delta_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,j}} - 2 \sum_{\substack{1 \le i < j \le k \\ i \neq \ell, \, j \neq \ell}} \frac{\nabla_{x_\ell} G_{N,\ell,i} \cdot \nabla_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,i} \, G_{N,\ell,j}}$$

We infer from (1.11) that $-\Delta G_N = -\frac{1}{2} N^{-1} V_N G_N$, so

$$-\frac{\Delta_{x_\ell} G_N^{(k)}}{G_N^{(k)}} = -\frac{1}{2N} \sum_{\substack{1 \le j \le k \\ j \neq \ell}} V_{N,\ell,j} - 2 \sum_{\substack{1 \le i < j \le k \\ i \neq \ell, \, j \neq \ell}} \frac{\nabla_{x_\ell} G_{N,\ell,i} \cdot \nabla_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,i} \, G_{N,\ell,j}}$$

Now summing in ℓ, $1 \le \ell \le k$, we obtain

$$H_N^{(k)} G_N^{(k)} = -2 G_N^{(k)} \sum_{\substack{1 \le i < j \le k \\ 1 \le \ell \le k \\ i \neq \ell, \, j \neq \ell}} \frac{\nabla_{x_\ell} G_{N,\ell,i} \cdot \nabla_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,i} \, G_{N,\ell,j}},$$

Here $H_N^{(k)} G_N^{(k)}$ is considered as $H_N^{(k)}$ applied to the function $G_N^{(k)}$. Note that the sum on the right side is perhaps more intuitively written as

$$H_N^{(k)} G_N^{(k)} = -G_N^{(k)} \sum_{\substack{1 \leq i,j,\ell \leq k \\ i,j,\ell \text{ distinct}}} \frac{\nabla_{x_\ell} G_{N,\ell,i} \cdot \nabla_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,i} \, G_{N,\ell,j}}$$

which implies (2.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

With the above Lemma, we compute the BBGKY hierarchy of $\left\{ \alpha_N^{(k)} \right\}$. Applying $Y_N^{(k)}$ to the left of the operator equation (2.2), we obtain

$$(2.5) \qquad\qquad Y_N^{(k)} H_N^{(k)} (Y_N^{(k)})^{-1} = (H_{N,0}^{(k)} + A_N^{(k)} + E_N^{(k)})$$

Thus $Y_N^{(k)}$ could be regarded as an approximation to the wave operator relating $H_N^{(k)}$ to $H_{N,0}^{(k)}$, although a more precise statement is that $Y_N^{(k)}$ is an exact wave operator relating $H_N^{(k)}$ to an approximation of $H_{N,0}^{(k)}$, namely the operator $H_{N,0}^{(k)} + A_N^{(k)} + E_N^{(k)}$. Since $H_{N,0}^{(k)} + A_N^{(k)} + E_N^{(k)}$ is not self-adjoint, the wave operator $Y_N^{(k)}$ is not unitary.

We now work out the BBGKY hierarchy of $\left\{ \alpha_N^{(k)} \right\}$. We will need to compute $Y_N^{(k)} [H_N^{(k)}, \gamma_N^{(k)}] Y_N^{(k)}$. To this end, we use the operator property: given two operators Y_1, Y_2, let $\alpha = Y_1 \gamma Y_2^{-1}$, then

$$Y_1 [H, \gamma] Y_2^{-1} = (Y_1 H Y_1^{-1}) \alpha - \alpha (Y_2 H Y_2^{-1}).$$

In the above, taking $Y_1 = Y_N^{(k)}$ and $Y_2 = (Y_N^{(k)})^{-1}$, and applying (2.5) give

$$(2.6) \qquad Y_N^{(k)} [H_N^{(k)}, \gamma_N^{(k)}] Y_N^{(k)} = (H_{N,0}^{(k)} + A_N^{(k)} + E_N^{(k)}) \alpha - \alpha (H_{N,0}^{(k)} + A_N^{(k)} + (E_N^{(k)})^*)$$

Moreover, let us introduce the operator $W_N^{(k)}$ which acts on any kernel $K(\mathbf{x}_k, \mathbf{x}_k')$ by

$$\begin{aligned} W_N^{(k)} K(\mathbf{x}_k, \mathbf{x}_k') &= [Y_N^{(k)} Y_N^{(k')} K](\mathbf{x}_k, \mathbf{x}_k') \\ &= \frac{1}{G_N(\mathbf{x}_k) G_N(\mathbf{x}_k')} K(\mathbf{x}_k, \mathbf{x}_k'). \end{aligned}$$

With the above notation, the BBGKY hierarchy of equations for the operators $\left\{ \alpha_N^{(k)} = Y_N^{(k)} \gamma_N^{(k)} Y_N^{(k)} \right\}$ or the corresponding kernels $\left\{ \alpha_N^{(k)}(\mathbf{x}_k, \mathbf{x}_k') = \frac{\gamma_N^{(k)}(\mathbf{x}_k, \mathbf{x}_k')}{G_N(\mathbf{x}_k) G_N(\mathbf{x}_k')} \right\}$ (using that $Y_N^{(k)}$ is equal to its transpose) is given by

$$(2.7) \qquad i\partial_t \alpha_N^{(k)} = \left(H_{N,0}^{(k)} - H_{N,0}^{(k')} \right) \alpha_N^{(k)} + \left(A_N^{(k)} - A_N^{(k')} \right) \alpha_N^{(k)} + \left(E_N^{(k)} - E_N^{(k')} \right) \alpha_N^{(k)}$$

$$+ \frac{N-k}{N} \sum_{l=1}^{k} W_N^{(k)} B_{N,l,k+1} (W_N^{(k+1)})^{-1} \alpha_N^{(k+1)}$$

where

$$(2.8) \qquad (W_N^{(k)} B_{N,l,k+1} (W_N^{(k+1)})^{-1} \alpha_N^{(k+1)})(\mathbf{x}_k; \mathbf{x}_k')$$

$$= \int_{\mathbb{R}^3} (V_N(x_l - x_{k+1}) - V_N(x_l' - x_{k+1})) \prod_{j=1}^{k} G_{N,j,k+1} G_{N,j',k+1} \alpha_N^{(k+1)}(...) dx_{k+1}$$

where $(\cdots)$ is $(x_1, \ldots, x_k, x_{k+1}; x'_1, \cdots, x'_k, x_{k+1})$.

We will decompose the terms in (2.8) to properly set up the Duhamel-Born series. Let

$$L_{N,\ell,k+1} + 1 \overset{\text{def}}{=} G_{N,\ell,k+1}^{-1} \prod_{j=1}^{k} G_{N,j,k+1} G_{N,j',k+1} = G_{N,\ell',k+1} \prod_{\substack{1 \le j \le k \\ j \ne \ell}} G_{N,j,k+1} G_{N,j',k+1},$$

$$L_{N,\ell',k+1} + 1 \overset{\text{def}}{=} G_{N,\ell',k+1}^{-1} \prod_{j=1}^{k} G_{N,j,k+1} G_{N,j',k+1} = G_{N,\ell,k+1} \prod_{\substack{1 \le j \le k \\ j \ne \ell}} G_{N,j,k+1} G_{N,j',k+1}.$$

Here L stands for localization. Also let

$$\tilde{V}_N(x) = V_N(x)(1 - w_N(x))$$

so that

$$\tilde{V}_N(x_l - x_{k+1}) = V_N(x_l - x_{k+1}) G_{N,\ell,k+1}$$
$$\tilde{V}_N(x'_l - x_{k+1}) = V_N(x'_l - x_{k+1}) G_{N,\ell',k+1}$$

Then

(2.9)
$$(W_N^{(k)} B_{N,l,k+1} (W_N^{(k+1)})^{-1} \alpha_N^{(k+1)})(\mathbf{x}_k; \mathbf{x}'_k)$$

$$= \int_{\mathbb{R}^3} \tilde{V}_N(x_l - x_{k+1}) (L_{N,l,k+1} + 1) \alpha_N^{(k+1)}(...) dx_{k+1}$$

$$- \int_{\mathbb{R}^3} \tilde{V}_N(x'_l - x_{k+1}) (L_{N,l',k+1} + 1) \alpha_N^{(k+1)}(...) dx_{k+1}$$

Separate "the k-body part" and "the 2-body part":

(2.10)
$$\tilde{B}_{N,\text{many}}^{(k+1)} \alpha_N^{(k+1)} = \sum_{l=1}^{k} \tilde{B}_{N,\text{many},l,k+1} \alpha_N^{(k+1)},$$

where

(2.11)
$$B_{N,\text{many},l,k+1} \alpha_N^{(k+1)} \overset{\text{def}}{=} \frac{N-k}{N} \int_{\mathbb{R}^3} (\tilde{V}_N(x_l - x_{k+1}) L_{N,l,k+1}$$
$$- \tilde{V}_N(x'_l - x_{k+1}) L_{N,l',k+1}) \alpha_N^{(k+1)}(\cdots) dx_{k+1}$$

and

(2.12)
$$\tilde{B}_N^{(k+1)} \alpha_N^{(k+1)}$$

$$\equiv \frac{N-k}{N} \sum_{l=1}^{k} \int_{\mathbb{R}^3} (\tilde{V}_N(x_l - x_{k+1}) - \tilde{V}_N(x'_l - x_{k+1})) \alpha_N^{(k+1)}(\cdots) dx_{k+1}$$

$$\equiv \sum_{l=1,}^{k} \tilde{B}_{N,l,k+1} \alpha_N^{(k+1)},$$

so that

$$\frac{N-k}{N} \sum_{\ell=1}^{k} W_N^{(k)} B_{N,\ell,k+1} (W_N^{(k+1)})^{-1} = \tilde{B}_{N,many}^{(k+1)} + \tilde{B}_N^{(k+1)}$$

The operator $\tilde{B}_{N,many}$ will give rise to the *k-body interaction part* and $\tilde{B}_N^{(k)}$ will give rise to the *interaction part* in the Duhamel-Born series below.

Finally, introduce the operator

$$\tilde{V}_N^{(k)}\alpha_N^{(k)} = (A_N^{(k)} - A_N^{(k)'})\alpha_N^{(k)} + (E_N^{(k)} - E_N^{(k)'})\alpha_N^{(k)}$$

which will give rise to the *potential part* in the Duhamel-Born series below.

From (2.7),

$$(2.13) \qquad \alpha_N^{(k)}(t_k) = U^{(k)}(t_k)\alpha_{N,0}^{(k)} - i\int_0^{t_k} U^{(k)}(t_k - t_{k+1})\tilde{V}_N^{(k)}\alpha_N^{(k)}(t_{k+1})\, dt_{k+1}$$

$$-i\int_0^{t_k} U^{(k)}(t_k - t_{k+1})\tilde{B}_{N,many}^{(k+1)}\alpha_N^{(k+1)}(t_{k+1})\, dt_{k+1}$$

$$-i\int_0^{t_k} U^{(k)}(t_k - t_{k+1})\tilde{B}_N^{(k+1)}\alpha_N^{(k+1)}(t_{k+1})\, dt_{k+1}$$

$$\equiv FP^{k,0} + PP^{k,0} + KIP^{k,0} + IP^{k,0},$$

Here, $U^{(k)}(t_k) = e^{it_k\triangle_{\mathbf{x}_j}}e^{-it_k\triangle_{\mathbf{x}_j'}}$, $FP^{k,0}$ stands for the free part of $\alpha_N^{(k)}$ with coupling level 0, $PP^{k,0}$ stands for the potential part of $\alpha_N^{(k)}$ with coupling level 0, $KIP^{k,0}$ stands for the k-body interaction part of $\alpha_N^{(k)}$ with coupling level 0, and $IP^{k,0}$ stands for the 2-body interaction part of $\alpha_N^{(k)}$ with coupling level 0. We will use this notation for the rest of the paper.

Remark 1. *In the case $\beta = 1$, $\tilde{B}_{N,l,k+1}$ is where $8\pi a_0$ shows up. In fact*

$$G_{N,\ell,k+1}V_{N,\ell,k+1} \rightarrow 8\pi a_0\delta(x_\ell - x_{k+1})$$

as shown in [33].

3. Proof of Theorem 1.1

We prove Theorem 1.1 as a warm up to the proof of Theorem 1.2. Here "warm up" means that we do not need to iterate (2.13) many times to get a good enough decay in time for the interaction part and do not need to use the Littlewood-Paley theory or the $X_{0,b}$ spaces.

To prove Theorem 1.1, we prove that hierarchy (2.13) converges to hierarchy (1.17) which written in the integral form is

$$(3.1)\quad \gamma^{(k)}(t_k) = U^{(k)}(t_k)\gamma_0^{(k)} - ic_0\int_0^{t_k} U^{(k)}(t_k - t_{k+1})\,\mathrm{Tr}_{k+1}\left[\delta(x_j - x_{k+1}), \gamma^{(k+1)}(t_{k+1})\right]dt_{k+1}.$$

It has been proven in [1, 28, 30, 33, 31, 32, 45, 11, 20] that, provided that the energy bound (1.16) holds, the 1st term and the last term on the right handside of (2.13) do converge to the right hand side of (3.1) weak*-ly in $L_T^\infty \mathcal{L}^1$. In particular, it is proved that, as trace class operators

$$\int_0^{t_k} U^{(k)}(t_k - t_{k+1})\tilde{B}_{N,j,k+1}\alpha_N^{(k+1)}(t_{k+1})\, dt_{k+1}$$

$$\rightharpoonup \left(\lim_{N\to\infty}\int \tilde{V}_N(x)dx\right)\int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_{j,k+1}\gamma^{(k+1)}(t_{k+1})\, dt_{k+1} \text{ weak*}$$

where $\lim_{N\to\infty} \int \tilde{V}_N(x)dx$ is exactly the c_0 defined in (1.18). So we only need to prove the following two estimates:

$$(3.2) \qquad\qquad \left\| PP^{k,0} \right\|_{L_T^\infty \mathcal{L}^2} \quad \to \quad 0 \text{ as } N \to \infty$$

$$(3.3) \qquad\qquad \left\| KIP^{k,0} \right\|_{L_T^\infty \mathcal{L}^2} \quad \to \quad 0 \text{ as } N \to \infty.$$

where

$$PP^{k,0}(t_k) \;=\; -i \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{V}_N^{(k)} \alpha_N^{(k)}(t_{k+1}) \, dt_{k+1},$$

$$KIP^{k,0}(t_k) \;=\; -i \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{B}_{N,many}^{(k+1)} \alpha_N^{(k+1)}(t_{k+1}) \, dt_{k+1}.$$

Before delving into the proof, we remark that condition (1.19) implies that

$$\sup_t \left(\left\| S^{(k+1)} \alpha_N^{(k+1)} \right\|_{L^2_{\mathbf{x}_k, \mathbf{x}'_k}} + \frac{1}{\sqrt{N}} \left\| S_1 S^{(k)} \alpha_N^{(k)} \right\|_{L^2_{\mathbf{x}_k, \mathbf{x}'_k}} \right) \leqslant C_0^{k+1}$$

In fact, consider the second term for $k = 2$:

$$\frac{1}{\sqrt{N}} \left\| S_1^2 S_2 S_{1'} S_{2'} \alpha_N^{(2)} \right\|_{L^2_{\mathbf{x}_1, \mathbf{x}'_1}}$$

$$= \frac{1}{\sqrt{N}} \left(\int \left| \int S_1^2 S_2 \left(\frac{\psi_N(\mathbf{x}_2, \mathbf{x}_{N-1})}{G_N^{(2)}(\mathbf{x}_2)} \right) S_{1'} S_{2'} \frac{\psi_N(\mathbf{x}'_2, \mathbf{x}_{N-2})}{G_N^{(k)}(\mathbf{x}'_2)} d\mathbf{x}_{N-2} \right|^2 d\mathbf{x}_2 \right)^{\frac{1}{2}}$$

Cauchy-Schwarz in $d\mathbf{x}_{N-2}$,

$$\leqslant \frac{1}{\sqrt{N}} \left(\int \left| S_1^2 S_2 \left(\frac{\psi_N(\mathbf{x}_2, \mathbf{x}_{N-1})}{G_N^{(2)}(\mathbf{x}_2)} \right) \right|^2 d\mathbf{x}_N \right)^{\frac{1}{2}} \left(\int \left| S_1 S_2 \frac{\psi_N(\mathbf{x}_2, \mathbf{x}_{N-2})}{G_N^{(k)}(\mathbf{x}_2)} \right|^2 d\mathbf{x}_N \right)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{N}} \left(\operatorname{Tr} S_1 S_{1'} S^{(2)} \alpha_N^{(2)} \right)^{\frac{1}{2}} \left(\operatorname{Tr} S^{(2)} \alpha_N^{(2)} \right)^{\frac{1}{2}}$$

$$\leqslant C_0^3$$

by condition (1.19) with $k = 2$.

3.1. Estimate for the Potential Term. Recall

$$\tilde{V}_N^{(k)} \alpha_N^{(k)} = (A_N^{(k)} - A_N^{(k)'}) \alpha_N^{(k)} + (E_N^{(k)} - E_N^{(k)'}) \alpha_N^{(k)},$$

where

$$A_N^{(k)} \alpha_N^{(k)} = - \sum_{\substack{1 \leq i,j,\ell \leq k \\ i,j,\ell \text{ distinct}}} \frac{\nabla_{x_\ell} G_{N,\ell,i} \cdot \nabla_{x_\ell} G_{N,\ell,j}}{G_{N,\ell,i} \, G_{N,\ell,j}} \alpha_N^{(k)},$$

and

$$E_N^{(k)} \alpha_N^{(k)} = 2 \sum_{\substack{1 \leq j,\ell \leq k \\ j \neq \ell}} \frac{\nabla_{x_\ell} G_{N,j,\ell}}{G_{N,j,\ell}} \cdot \nabla_{x_\ell} \alpha_N^{(k)}.$$

Let us define

$$(3.4) \qquad A_{N,i,j,l}^{(k)}\alpha_N^{(k)} = -\frac{\nabla_{x_\ell}G_{N,\ell,i} \cdot \nabla_{x_\ell}G_{N,\ell,j}}{G_{N,\ell,i}\,G_{N,\ell,j}}\alpha_N^{(k)},$$

$$(3.5) \qquad E_{N,j,l}^{(k)}\alpha_N^{(k)} = 2\frac{\nabla_{x_\ell}G_{N,j,\ell}}{G_{N,j,\ell}} \cdot \nabla_{x_\ell}\alpha_N^{(k)},$$

then to prove estimate (3.2), it suffices to prove the following estimates

$$\left\|\int_0^{t_k} U^{(k)}(t_k - t_{k+1})A_{N,i,j,l}^{(k)}\alpha_N^{(k)}\,(t_{k+1})\,dt_{k+1}\right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C_T N^{-2+},$$

$$\left\|\int_0^{t_k} U^{(k)}(t_k - t_{k+1})E_{N,j,l}^{(k)}\alpha_N^{(k)}\,(t_{k+1})\,dt_{k+1}\right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C_T N^{-1+}.$$

In fact, assume the above estimates for the moment, we have

$$\left\|PP^{k,0}\right\|_{L_T^\infty \mathcal{L}^2} = \left\|PP^{k,0}\right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C_T k^3 N^{-2+} + C_T k^2 N^{-1+}$$

$$\to \quad 0 \text{ as } N \to \infty,$$

for $\beta \in (0,1]$, where we used the facts that $\|\cdot\|_{L_T^\infty \mathcal{L}^2} = \|\cdot\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}$ and there are k^3 summands in $A_N^{(k)}$ while there are k^2 summands in $E_N^{(k)}$. So we finish the estimate for the potential part in the proof of Theorem 1.1 with the following two lemmas.

Lemma 3.1. *We have the estimate:*

$$\left\|\int_0^{t_k} U^{(k)}(t_k - t_{k+1})A_{N,i,j,l}^{(k)}\alpha^{(k)}\,(t_{k+1})\,dt_{k+1}\right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant \quad C_T N^{-2+}\left\|\langle\nabla_{x_i}\rangle\langle\nabla_{x_j}\rangle\langle\nabla_{x_l}\rangle\,\alpha^{(k)}\,(t_{k+1})\right\|_{L_t^\infty L_{\mathbf{x},\mathbf{x}'}^2}.$$

In particular, if one assumes the energy bound (1.16), it reads

$$\left\|\int_0^{t_k} U^{(k)}(t_k - t_{k+1})A_{N,i,j,l}^{(k)}\alpha_N^{(k)}\,(t_{k+1})\,dt_{k+1}\right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C_T N^{-2+}.$$

Lemma 3.2.

$$\left\|\int_0^{t_k} U^{(k)}(t_k - t_{k+1})E_{N,j,l}^{(k)}\alpha^{(k)}\,(t_{k+1})\,dt_{k+1}\right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant \quad C_T N^{-1+}\left\|\langle\nabla_{x_j}\rangle\langle\nabla_{x_\ell}\rangle\,\alpha^{(k)}\,(t_{k+1})\right\|_{L_t^\infty L_{\mathbf{x},\mathbf{x}'}^2}.$$

In particular, if one assumes the energy bound (1.16), it reads

$$\left\|\int_0^{t_k} U^{(k)}(t_k - t_{k+1})E_{N,j,l}^{(k)}\alpha_N^{(k)}\,(t_{k+1})\,dt_{k+1}\right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C_T N^{-1+}.$$

Proof of Lemma 3.1. Define

$$v_{2,N}(x) = \frac{N^{2\beta-1}\,(\nabla\omega_0)\,(N^\beta x)}{G_N(x)},$$

then

$$v_{2,N}(x) = \frac{N^{2\beta-1}\left(\nabla\omega_0\right)\left(N^\beta x\right)}{G_N(x)} \sim N^{2\beta-1}\left\langle N^\beta x\right\rangle^{-2} \equiv \tilde{v}_{2,N}(x).$$

So

$$\left\|\int_0^{t_k} U^{(k)}(t_k - t_{k+1})A^{(k)}_{N,i,j,l}\alpha^{(k)}\left(t_{k+1}\right) dt_{k+1}\right\|_{L_T^\infty L^2_{\mathbf{x},\mathbf{x}'}}$$

$$\sim \left\|\int_0^{t_k} U^{(k)}(t_k - t_{k+1})\left[\tilde{v}_{2,N}(x_l - x_i)\tilde{v}_{2,N}(x_l - x_j)\alpha^{(k)}\left(t_{k+1}\right)\right] dt_{k+1}\right\|_{L_T^\infty L^2_{\mathbf{x},\mathbf{x}'}}$$

Insert a smooth cut-off $\theta(t)$ with $\theta(t) = 1$ for $t \in [-T,T]$ and $\theta(t) = 0$ for $t \in [-2T, 2T]^c$ into the above,

$$\leqslant \left\|\theta(t_k)\int_0^{t_k} U^{(k)}(t_k - t_{k+1})\left[\tilde{v}_{2,N}(x_l - x_i)\tilde{v}_{2,N}(x_l - x_j)\theta(t_{k+1})\alpha^{(k)}\left(t_{k+1}\right)\right] dt_{k+1}\right\|_{L_t^\infty L^2_{\mathbf{x},\mathbf{x}'}}$$

Since $\|\cdot\|_{L_t^\infty L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C\|\cdot\|_{X^{(k)}_{\frac{1}{2}+}}$, we have

$$\leqslant C\left\|\theta(t_k)\int_0^{t_k} U^{(k)}(t_k - t_{k+1})\left[\tilde{v}_{2,N}(x_l - x_i)\tilde{v}_{2,N}(x_l - x_j)\theta(t_{k+1})\alpha^{(k)}\left(t_{k+1}\right)\right] dt_{k+1}\right\|_{X^{(k)}_{\frac{1}{2}+}}.$$

By Lemma 5.1,

$$\leqslant C\left\|\tilde{v}_{2,N}(x_l - x_i)\tilde{v}_{2,N}(x_l - x_j)\theta(t_{k+1})\alpha^{(k)}\left(t_{k+1}\right)\right\|_{X^{(k)}_{-\frac{1}{2}+}}.$$

Use the first inequality of (5.18) in Corollary 5.10,

$$\leqslant C\left\|\tilde{v}_{2,N}\right\|^2_{L^{\frac{3}{2}+}}\left\|\theta(t_{k+1})\left\langle\nabla_{x_i}\right\rangle\left\langle\nabla_{x_j}\right\rangle\left\langle\nabla_{x_l}\right\rangle\alpha^{(k)}\left(t_{k+1}\right)\right\|_{L_t^2 L^2_{\mathbf{x},\mathbf{x}'}}$$

$$\leqslant C_T\left\|\tilde{v}_{2,N}\right\|^2_{L^{\frac{3}{2}+}}\left\|\left\langle\nabla_{x_i}\right\rangle\left\langle\nabla_{x_j}\right\rangle\left\langle\nabla_{x_l}\right\rangle\alpha^{(k)}\left(t_{k+1}\right)\right\|_{L_t^\infty L^2_{\mathbf{x},\mathbf{x}'}}$$

where

$$\left\|\tilde{v}_{2,N}\right\|_{L^{\frac{3}{2}+}} = CN^{-1+}.$$

That is

$$\left\|\int_0^{t_k} U^{(k)}(t_k - t_{k+1})A^{(k)}_{N,i,j,l}\alpha^{(k)}\left(t_{k+1}\right) dt_{k+1}\right\|_{L_T^\infty L^2_{\mathbf{x},\mathbf{x}'}}$$

$$\leqslant C_T N^{-2+}\left\|\left\langle\nabla_{x_i}\right\rangle\left\langle\nabla_{x_j}\right\rangle\left\langle\nabla_{x_l}\right\rangle\alpha^{(k)}\left(t_{k+1}\right)\right\|_{L_t^\infty L^2_{\mathbf{x},\mathbf{x}'}}$$

as claimed. □

Proof of Lemma 3.2. As in the proof of Lemma 3.1, we replace

$$v_{2,N}(x) = \frac{N^{2\beta-1}\left(\nabla\omega_0\right)\left(N^\beta x\right)}{G_N(x)} \sim N^{2\beta-1}\left\langle N^\beta x\right\rangle^{-2} = \tilde{v}_{2,N}(x)$$

with $\tilde{v}_{2,N}(x) = N^{2\beta-1} \langle N^\beta x \rangle^{-2}$. Then, we have

$$\left\| \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) E_{N,j,l}^{(k)} \alpha^{(k)}(t_{k+1}) \, dt_{k+1} \right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\sim \left\| \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \left[\tilde{v}_{2,N}(x) \left(|\nabla_{x_\ell}| \alpha^{(k)}(t_{k+1}) \right) \right] dt_{k+1} \right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant \left\| \tilde{v}_{2,N}(x) \left(\theta(t_{k+1}) |\nabla_{x_\ell}| \alpha^{(k)}(t_{k+1}) \right) \right\|_{X_{-\frac{1}{2}+}^{(k)}}$$

Use the second inequality of (5.6) in Corollary 5.6,

$$\leqslant C \left\| \tilde{v}_{2,N} \right\|_{L^{\frac{3}{2}+}} \left\| \theta(t_{k+1}) \left| \nabla_{x_j} \right| |\nabla_{x_\ell}| \alpha^{(k)}(t_{k+1}) \right\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant C_T N^{-1+} \left\| \langle \nabla_{x_j} \rangle \langle \nabla_{x_\ell} \rangle \alpha^{(k)}(t_{k+1}) \right\|_{L_t^\infty L_{\mathbf{x},\mathbf{x}'}^2}.$$

So we have finished the proof of Lemma 3.2. $\square$

3.2. Estimate for the k-body Interaction Part. Recall

$$\tilde{B}_{N,many}^{(k+1)} \alpha_N^{(k+1)} = \sum_{l=1}^k \tilde{B}_{N,many,l,k+1} \alpha_N^{(k+1)}.$$

To prove estimate (3.3), we prove the estimate:

$$(3.6) \qquad \left\| \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{B}_{N,many,l,k+1}^+ \alpha_N^{(k+1)}(t_{k+1}) \, dt_{k+1} \right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C_T C^{k+1} N^{-\frac{1}{2}+},$$

where $\tilde{B}_{N,many,l,k+1}^+$ is half of $\tilde{B}_{N,many,l,k+1}$. Assume estimate (3.6), then

$$\left\| KIP^{k,0} \right\|_{L_T^\infty \mathcal{L}^2} = \left\| KIP^{k,0} \right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2} \leqslant 2k C_T C^{k+1} N^{-\frac{1}{2}+} \to 0 \text{ as } N \to \infty.$$

The rest of this section is the proof of estimate (3.6). We first give the following lemma.

Lemma 3.3. *One can decompose $\tilde{B}_{N,many,l,k+1}^+ \alpha_N^{(k+1)}(t_{k+1})$, defined in (2.11), as the sum of at most 8^k terms of the form*

$$\tilde{B}_{N,many,l,k+1,\sigma}^+ \alpha_N^{(k+1)}(t_{k+1})$$
$$= \frac{N-k}{N} \int_{\mathbb{R}^3} \tilde{V}_N(x_l - x_{k+1}) N^{\beta-1} w_0(N^\beta(x_\sigma - x_{k+1})) A_\sigma \alpha_N^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}_k', x_{k+1}) \, dx_{k+1}.$$

Here, x_σ is some x_j or x_j' but not x_l and A_σ is a product of $\left[N^{\beta-1} w_0(N^\beta(x_j - x_{k+1})) \right]$, $\left[N^{\beta-1} w_0(N^\beta(x_j' - x_{k+1})) \right]$ or 1 with x_j not equal to x_l or x_σ.

Proof. Recall,

$$\tilde{B}_{N,many,l,k+1}^+ \alpha_N^{(k+1)} = \frac{N-k}{N} \int_{\mathbb{R}^3} \tilde{V}_N(x_l - x_{k+1}) L_{N,l,k+1}$$
$$\alpha_N^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}_k', x_{k+1}) \, dx_{k+1}$$

Notice that,

$$L_{N,l,k+1} + 1 \quad = \quad G_{N,l',k+1} \prod_{\substack{1 \leqslant j \leqslant k \\ j \neq l}} G_{N,j,k+1} G_{N,j',k+1}$$

$$G_{N,j,k+1} \quad = \quad 1 - N^{\beta-1} w_0(N^\beta(x_j - x_{k+1}))$$

Thus, taken as a binomial expansion, $L_{N,l,k+1}$ is a sum of $2k$ classes where each class has $\binom{2k}{j}, j = 1, ..., 2k$, terms inside, that is:

$$L_{N,l,k+1}$$

$$= \quad \left(\sum_{\substack{1 \leqslant j \leqslant k \\ j \neq l}} N^{\beta-1} w_0(N^\beta(x_j - x_{k+1})) + \sum_{1 \leqslant j \leqslant k} N^{\beta-1} w_0(N^\beta(x_j' - x_{k+1})) \right)$$

$$+ ...$$

$$+ ([N^{\beta-1} w_0(N^\beta(x_l' - x_{k+1}))] \prod_{\substack{1 \leqslant j \leqslant k \\ j \neq l}} [N^{\beta-1} w_0(N^\beta(x_j - x_{k+1}))] [N^{\beta-1} w_0(N^\beta(x_j' - x_{k+1}))]).$$

Thus $L_{N,l,k+1}$ can be written as a sum of at most 8^k terms which individually looks like

$$N^{\beta-1} w_0(N^\beta(x_\sigma - x_{k+1})) A_\sigma$$

where x_σ is some x_j or x_j' but not x_l and A_σ is a product of $[N^{\beta-1} w_0(N^\beta(x_j - x_{k+1}))]$, $[N^{\beta-1} w_0(N^\beta(x_j' - x_{k+1}))]$ or 1 with x_j not equal to x_l or x_σ. Inserting this into (2.11), we have the claimed decomposition. $\qquad\square$

With Lemma 3.3, we have the following estimate.

Lemma 3.4. *Let $\tilde{B}_{N,many,l,k+1,\sigma}^+$ be defined as in Lemma 3.3, we have*

$$\left\| \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{B}_{N,many,l,k+1,\sigma}^+ \alpha^{(k+1)}(t_{k+1}) \, dt_{k+1} \right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant \quad C_T C^{k+1} N^{-\frac{1}{2}+} \frac{1}{\sqrt{N}} \left\| \langle \nabla_{x_{k+1}} \rangle^2 \left\langle \nabla_{x_{k+1}'} \right\rangle \alpha^{(k+1)}(t_{k+1}) \right\|_{L_t^\infty L_{\mathbf{x},\mathbf{x}'}^2}.$$

Consequently,

$$\left\| \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{B}_{N,many,l,k+1}^+ \alpha^{(k+1)}(t_{k+1}) \, dt_{k+1} \right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant \quad C_T C^{k+1} N^{-\frac{1}{2}+} \frac{1}{\sqrt{N}} \left\| \langle \nabla_{x_{k+1}} \rangle^2 \left\langle \nabla_{x_{k+1}'} \right\rangle \alpha^{(k+1)}(t_{k+1}) \right\|_{L_t^\infty L_{\mathbf{x},\mathbf{x}'}^2}$$

because, by Lemma 3.3,

$$\tilde{B}_{N,many,l,k+1}^+ \alpha_N^{(k+1)} = \sum_\sigma \tilde{B}_{N,loc,l,k+1,\sigma}^+,$$

where the sum has at most 8^k terms inside. In particular, if one assumes the energy bound (1.16), it reads

$$\left\| \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{B}^+_{N,many,l,k+1} \alpha^{(k+1)}(t_{k+1}) \, dt_{k+1} \right\|_{L_T^\infty L^2_{\mathbf{x},\mathbf{x}'}}$$

$$\leqslant \quad C_T C^{k+1} N^{-\frac{1}{2}+},$$

which is exactly estimate (3.6).

Proof. Recall

$$\tilde{B}^+_{N,many,l,k+1,\sigma} \alpha^{(k+1)}(t_{k+1})$$
$$= \quad \frac{N-k}{N} \int_{\mathbb{R}^3} \tilde{V}_N(x_l - x_{k+1}) N^{\beta-1} w_0(N^\beta(x_\sigma - x_{k+1})) A_\sigma \alpha^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) dx_{k+1}.$$

There is no need to write out the variables in A_σ. In fact, A_σ is a harmless factor because $N^{\beta-1} w_0(N^\beta(x_j - x_{k+1}))$ is in L^∞ uniformly in N if $\beta \leqslant 1$.

As in the proof of Lemmas 3.1 and 3.2, we insert a smooth cut-off $\theta(t)$,

$$\left\| \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{B}^+_{N,many,l,k+1,\sigma} \alpha^{(k+1)}(t_{k+1}) \, dt_{k+1} \right\|_{L_T^\infty L^2_{\mathbf{x},\mathbf{x}'}}$$

$$\leqslant \quad \left\| \theta(t_k) \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{B}^+_{N,many,l,k+1,\sigma} \left(\theta(t_{k+1}) \alpha^{(k+1)}(t_{k+1}) \right) dt_{k+1} \right\|_{L_t^\infty L^2_{\mathbf{x},\mathbf{x}'}},$$

and proceed to

$$\leqslant \quad C \left\| \tilde{B}^+_{N,many,l,k+1,\sigma} \left(\theta(t_{k+1}) \alpha^{(k+1)}(t_{k+1}) \right) \right\|_{X^{(k)}_{-\frac{1}{2}+}}$$

$$= \quad C \left\| \int_{\mathbb{R}^3} \tilde{V}_N(x_l - x_{k+1}) N^{\beta-1} w_0(N^\beta(x_\sigma - x_{k+1})) \right.$$

$$\left. A_\sigma \theta(t_{k+1}) \alpha^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) dx_{k+1} \right\|_{X^{(k)}_{-\frac{1}{2}+}}.$$

The third inequality of (5.21) of Lemma 5.12 gives

$$\leqslant \quad C \left\| \tilde{V}_N \right\|_{L^{1+}} \left\| N^{\beta-1} w_0(N^\beta(\cdot)) \right\|_{L^{3+}} \left\| A_\sigma \right\|_{L^\infty}$$

$$\times \left\| \langle \nabla_{x_{k+1}} \rangle^2 \langle \nabla_{x'_{k+1}} \rangle \theta(t_{k+1}) \alpha^{(k+1)}(\mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) \right\|_{L_t^2 L^2_{\mathbf{x},\mathbf{x}'}}$$

where

$$\left\| \tilde{V}_N \right\|_{L^{1+}} \left\| N^{\beta-1} w_0(N^\beta(\cdot)) \right\|_{L^{3+}} \left\| A_\sigma \right\|_{L^\infty} \leqslant C^{k+1} N^{-1+}.$$

Thus

$$\left\| \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{B}_{N,many,l,k+1,\sigma}^+ \alpha^{(k+1)}(t_{k+1}) \, dt_{k+1} \right\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant \quad C_T C^{k+1} N^{-\frac{1}{2}+} \frac{1}{\sqrt{N}} \left\| \langle \nabla_{x_{k+1}} \rangle^2 \left\langle \nabla_{x'_{k+1}} \right\rangle \alpha^{(k+1)}(t_{k+1}) \right\|_{L_t^\infty L_{\mathbf{x},\mathbf{x}'}^2}.$$

which is good enough to conclude the proof of Lemma 3.4.   □

4. PROOF OF THEOREM 1.2

We will use Littlewood-Paley theory to prove Theorem 1.2. Let $P_{\leq M}^i$ be the projection onto frequencies $\leq M$ and P_M^i the analogous projections onto frequencies $\sim M$, acting on functions of $x_i \in \mathbb{R}^3$ (the ith coordinate). We take M to be a dyadic frequency range $2^\ell \geq 1$. Similarly, we define $P_{\leq M}^{i'}$ and $P_M^{i'}$, which act on the variable x_i'. Let

$$(4.1) \qquad\qquad P_{\leq M}^{(k)} = \prod_{i=1}^k P_{\leq M}^i P_{\leq M}^{i'}.$$

As observed in earlier work [13, 19, 21], to establish Theorem 1.2, it suffices to prove the following theorem.[6]

Theorem 4.1. *Under the assumptions of Theorem 1.2, there exists a C (independent of k, M, N) such that for each $M \geqslant 1$ there exists N_0 (depending on M) such that for $N \geqslant N_0$, there holds*

$$(4.2) \qquad\qquad \| P_{\leq M}^{(k)} R^{(k)} \tilde{B}_{N,j,k+1} \gamma_N^{(k+1)}(t) \|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C^k.$$

In fact, passing to the weak limit $\gamma_N^{(k)} \to \gamma^{(k)}$ as $N \to \infty$, we obtain

$$\| P_{\leq M}^{(k)} R^{(k)} B_{j,k+1} \gamma^{(k+1)} \|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C^k$$

Since it holds uniformly in M, we can send $M \to \infty$ and, by the monotone convergence theorem, we obtain

$$\| R^{(k)} B_{j,k+1} \gamma^{(k+1)} \|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C^k$$

which is exactly the Klainerman-Machedon space-time bound (1.20). This completes the proof Theorem 1.2, assuming Theorem 4.1.

The rest of this section is devoted to proving Theorem 4.1. We will first establish estimate (4.2) for a sufficiently small T which depends on the controlling constant in condition (1.19) and is independent of k, N and M, then a bootstrap argument together with condition (1.19) give estimate (4.2) for every finite time at the price of a larger constant C. The first step of the proof of Theorem 4.1 is to iterate (2.13) p times and get to the formula

$$\alpha_N^{(k)}(t_k) = FP^{k,p}(t_k) + PP^{k,p}(t_k) + KIP^{k,p}(t_k) + IP^{k,p}(t_k),$$

---

[6]To be precise, this formulation with frequency localization is from [21]. The formulations in [13, 19] do not have the Littlewood-Paley projector inside.

then estimate each term, that is, prove the following estimates:

$$(4.3) \qquad \left\| P_{\leq M}^{(k-1)} R^{(k-1)} \tilde{B}_{N,1,k} FP^{k,p} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C^{k-1},$$

$$(4.4) \qquad \left\| P_{\leq M}^{(k-1)} R^{(k-1)} \tilde{B}_{N,1,k} PP^{k,p} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C^{k-1},$$

$$(4.5) \qquad \left\| P_{\leq M}^{(k-1)} R^{(k-1)} \tilde{B}_{N,1,k} KIP^{k,p} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C^{k-1},$$

$$(4.6) \qquad \left\| P_{\leq M}^{(k-1)} R^{(k-1)} \tilde{B}_{N,1,k} IP^{k,p} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C^{k-1}.$$

for all $k \geqslant 2$ and for some C and a sufficiently small T determined by the controlling constant in condition (1.19) and independent of k, N and M. Here, we iterate (2.13) because it is difficult to show (4.6) unless $p = \ln N$, a fact first observed by Chen and Pavlovic [13], who proved (1.20) for $\beta \in (0, 1/4)$, and then used in the $\beta \in (0, 2/7)$ work [19] by X.C and in the $\beta \in (0, 2/3)$ work [21] by X.C and J.H. As proven in [19, 21], once p is set to be $\ln N$, one can prove estimates (4.3) and (4.6) for all $\beta \in (0, \infty)$. The obstacle in achieving higher β lies solely in proving (4.4) and (4.5). Hence, in the rest of this section, we prove estimates (4.4) and (4.5) only and refer the readers to [19, 21] for the proof of estimates (4.3) and (4.6).

To make formulas shorter, for $q \geqslant 1$, we introduce the following notation:

$$J_N^{(k,q)}(\underline{t}_{k,q})(f^{(k+q)}) = \left(U^{(k)}(t_k - t_{k+1}) \tilde{B}_N^{(k+1)} \right) \cdots \left(U^{(k+q-1)}(t_{k+q-1} - t_{k+q}) \tilde{B}_N^{(k+q)} \right) f^{(k+q)},$$

where $\underline{t}_{k,q}$ means $(t_{k+1}, \ldots, t_{k+q})$. When $q = 0$, the above product is degenerate and we let

$$J_N^{(k,0)}(\underline{t}_{k,0})(f^{(k)}) = f^{(k)}.$$

Now plug the $(k+1)$ version of (2.13) into the *last term only* of (2.13) to obtain

$$\alpha_N^{(k)}(t_k) = FP^{k,1}(t_k) + PP^{k,1}(t_k) + KIP^{k,1}(t_k) + IP^{k,1}(t_k)$$

where the *free part* is

$$\begin{aligned} &FP^{k,1}(t_k) \\ =\ & U^{(k)}(t_k)\alpha_{N,0}^{(k)} + (-i) \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{B}_N^{(k+1)} U^{(k+1)}(t_{k+1})\alpha_{N,0}^{(k+1)} \, dt_{k+1} \\ =\ & \sum_{q=0}^{1} (-i)^q \int_{0 \leq t_{k+q-1} \leq \cdots \leq t_k} J_N^{(k,q)}(\underline{t}_{k,q})(f_{FP}^{(k,q)}) dt_{\underline{k,q}} \end{aligned}$$

with

$$(4.7) \qquad f_{FP}^{(k,q)} = U^{(k+q)}(t_{k+q})\alpha_{N,0}^{(k+q)}$$

the *potential part* is

$$
PP^{k,1}(t_k)
$$
$$
= -i \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{V}_N^{(k)} \alpha_N^{(k)}(t_{k+1}) \, dt_{k+1}
$$
$$
+ (-i)^2 \int_0^{t_k} \int_0^{t_{k+1}} U^{(k)}(t_k - t_{k+1}) \tilde{B}_N^{(k+1)} U^{(k+1)}(t_{k+1} - t_{k+2}) \tilde{V}_N^{(k+1)} \alpha_N^{(k+1)}(t_{k+2}) \, d\underline{t}_{k,2}
$$
$$
= \sum_{q=0}^{1} (-i)^{q+1} \int_{0 \leq t_{k+q-1} \leq \cdots \leq t_k} J_N^{(k,q)}(\underline{t}_{k,q})(f_{PP}^{(k,q)}) d\underline{t}_{k,q}
$$

with

$$
(4.8) \qquad f_{PP}^{(k,q)} = \int_0^{t_{k+q}} U^{(k+q)}(t_{k+q} - t_{k+q+1}) \tilde{V}_N^{(k+q)} \alpha_N^{(k+q)}(t_{k+q+1}) \, dt_{k+q+1}
$$

the *k-body interaction part* is

$$
KIP^{k,1}(t_k)
$$
$$
= -i \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \tilde{B}_{N,many}^{(k+1)} \alpha_N^{(k+1)}(t_{k+1}) \, dt_{k+1}
$$
$$
+ (-i)^2 \int_0^{t_k} \int_0^{t_{k+1}} U^{(k)}(t_k - t_{k+1}) \tilde{B}_N^{(k+1)} U^{(k+1)}(t_{k+1} - t_{k+2}) \tilde{B}_{N,many}^{(k+2)} \alpha_N^{(k+2)}(t_{k+2}) \, d\underline{t}_{k,2}
$$
$$
= \sum_{q=0}^{1} (-i)^{q+1} \int_{0 \leq t_{k+q-1} \leq \cdots \leq t_k} J_N^{(k,q)}(\underline{t}_{k,q})(f_{KIP}^{(k,q)}) d\underline{t}_{k,q}
$$

with

$$
(4.9) \qquad f_{KIP}^{(k,q)} = \int_0^{t_{k+q}} U^{(k+q)}(t_{k+q} - t_{k+q+1}) \tilde{B}_{N,many}^{(k+q+1)} \alpha_N^{(k+q+1)}(t_{k+q+1}) \, dt_{k+q+1}
$$

and the *interaction part* is

$$
IP^{k,1}(t_k)
$$
$$
= (-i)^2 \int_0^{t_k} \int_0^{t_{k+1}} U^{(k)}(t_k - t_{k+1}) \tilde{B}_N^{(k+1)} U^{(k+1)}(t_{k+1} - t_{k+2}) \tilde{B}_N^{(k+2)} \alpha_N^{(k+2)}(t_{k+2}) \, d\underline{t}_{k,2}
$$
$$
= (-i)^{1+1} \int_0^{t_k} \int_0^{t_{k+1}} J_N^{(k,2)}(\underline{t}_{k,2})(\alpha_N^{(k+2)}(t_{k+2})) d\underline{t}_{k,2}
$$

Now we iterate this process $(p-1)$ more times to obtain

$$
\alpha_N^{(k)}(t_k) = FP^{k,p}(t_k) + PP^{k,p}(t_k) + KIP^{k,p}(t_k) + IP^{k,p}(t_k)
$$

where the *free part* is

$$
(4.10) \qquad FP^{k,p}(t_k) = \sum_{q=0}^{p} (-i)^q \int_{0 \leq t_{k+q-1} \leq \cdots \leq t_k} J_N^{(k,q)}(\underline{t}_{k,q})(f_{FP}^{(k,q)}) d\underline{t}_{k,q}.
$$

The *potential part* is

$$(4.11) \qquad PP^{k,p}(t_k) = \sum_{q=0}^{p} (-i)^{q+1} \int_{0 \leq t_{k+q-1} \leq \cdots \leq t_k} J_N^{(k,q)}(\underline{t}_{k,q})(f_{PP}^{(k,q)})d\underline{t}_{k,q}.$$

The *k-body interaction part* is

$$(4.12) \qquad KIP^{k,p}(t_k) = \sum_{q=0}^{p} (-i)^{q+1} \int_{0 \leq t_{k+q-1} \leq \cdots \leq t_k} J_N^{(k,q)}(\underline{t}_{k,q})(f_{KIP}^{(k,q)})d\underline{t}_{k,q}.$$

The *interaction part* is

$$(4.13) \qquad IP^{k,p}(t_k) = (-i)^{p+1} \int_{0 \leq t_{k+p} \leq \cdots \leq t_k} J_N^{(k+p+1)}(\underline{t}_{k,p+1})(\alpha_N^{(k+p+1)}(t_{k+p+1}))d\underline{t}_{k,p+1}.$$

We then apply the Klainerman-Machedon board game to the free part, potential part, k-body interaction part, and interaction part.

Lemma 4.2 (Klainerman-Machedon board game). [47] *One can express*

$$\int_{0 \leq t_{k+q-1} \leq \cdots \leq t_k} J_N^{(k,q)}(\underline{t}_{k,q})(f^{(k+q)})d\underline{t}_{k,q},$$

as a sum of at most 4^q terms of the form

$$\int_D J_N^{(k,q)}(\underline{t}_{k,q}, \mu_m)(f^{(k+q)})d\underline{t}_{k,q},$$

or in other words,

$$\int_{0 \leq t_{k+q-1} \leq \cdots \leq t_k} J_N^{(k,q)}(\underline{t}_{k,q})(f^{(k+q)})d\underline{t}_{k,q} = \sum_m \int_D J_N^{(k,q)}(\underline{t}_{k,q}, \mu_m)(f^{(k+q)})d\underline{t}_{k,q}.$$

Here $D \subset [0, t_k]^q$, μ_m are a set of maps from $\{k+1, \ldots, k+q\}$ to $\{k, \ldots, k+q-1\}$ satisfying $\mu_m(k+1) = k$ and $\mu_m(l) < l$ for all l, and

$$\begin{aligned} J_N^{(k,q)}(\underline{t}_{k,q}, \mu_m)(f^{(k+q)}) &= U^{(k)}(t_k - t_{k+1})\tilde{B}_{N,k,k+1}U^{(k+1)}(t_{k+1} - t_{k+2})\tilde{B}_{N,\mu_m(k+2),k+2} \cdots \\ &\quad \cdots U^{(k+q-1)}(t_{k+q-1} - t_{k+q})\tilde{B}_{N,\mu_m(k+q),k+q}(f^{(k+q)}). \end{aligned}$$

4.1. **Estimate for the k-body Interaction Part.** To make formulas shorter, let us write

$$R_{\leqslant M_k}^{(k)} = P_{\leqslant M_k}^{(k)} R^{(k)},$$

since $P_{\leqslant M_k}^{(k)}$ and $R^{(k)}$ are usually bundled together.

4.1.1. *Step I.* Applying Lemma 4.2 to (4.12), we get

$$\left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} KIP^{k,p} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant \sum_{q=0}^{p} \sum_m \left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} \int_D J_N^{(k,q)}(\underline{t}_{k,q}, \mu_m)(f_{KIP}^{(k,q)})d\underline{t}_{k,q} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$

where $f_{KIP}^{(k,q)}$ is given by (4.9) and the sum $\sum_m$ has at most 4^q terms inside. By Minkowski's integral inequality,

$$\left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} \int_D J_N^{(k,q)}(\underline{t}_{k,q}, \mu_m)(f_{KIP}^{(k,q)}) d\underline{t}_{k,q} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$= \int_0^T dt_k \left\| \int_D R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} J_N^{(k,q)}(\underline{t}_{k,q}, \mu_m)(f_{KIP}^{(k,q)}) d\underline{t}_{k,q} \right\|_{L_{\mathbf{x},\mathbf{x}'}^2} dt_k$$

$$\leqslant \int_{[0,T]^{q+1}} \left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} U^{(k)}(t_k - t_{k+1}) \tilde{B}_{N,k,k+1} \cdots \right\|_{L_{\mathbf{x},\mathbf{x}'}^2} dt_k d\underline{t}_{k,q}.$$

Cauchy-Schwarz in the t_k integration,

$$\leqslant T^{\frac{1}{2}} \int_{[0,T]^q} \left(\int \left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} U^{(k)}(t_k - t_{k+1}) \tilde{B}_{N,k,k+1} \cdots \right\|_{L_{\mathbf{x},\mathbf{x}'}^2}^2 dt_k \right)^{\frac{1}{2}} d\underline{t}_{k,q}$$

By Lemma 5.2,

$$\leqslant C_\varepsilon T^{\frac{1}{2}} \sum_{M_k \geqslant M_{k-1}} \left(\frac{M_{k-1}}{M_k} \right)^{1-\varepsilon} \int_{[0,T]^q} \left\| R_{\leqslant M_k}^{(k)} \tilde{B}_{N,k,k+1} U^{(k+1)}(t_{k+1} - t_{k+2}) \cdots \right\|_{L_{\mathbf{x},\mathbf{x}'}^2} d\underline{t}_{k,q}$$

Iterate the previous steps $(q-1)$ times,

$$\leqslant (C_\varepsilon T^{\frac{1}{2}})^q \sum_{M_{k+q-1} \geqslant \cdots \geqslant M_k \geqslant M_{k-1}} \left[\left(\frac{M_{k-1}}{M_k} \frac{M_k}{M_{k+1}} \cdots \frac{M_{k+q-2}}{M_{k+q-1}} \right)^{1-\varepsilon} \right.$$

$$\left. \times \left\| R_{\leqslant M_{k+q-1}}^{(k+q-1)} B_{N,\mu_m(k+q),k+q} \left(f_{KIP}^{(k,q)} \right) \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \right]$$

$$= (C_\varepsilon T^{\frac{1}{2}})^q \sum_{M_{k+q-1} \geqslant \cdots \geqslant M_k \geqslant M_{k-1}} \left[\left(\frac{M_{k-1}}{M_{k+q-1}} \right)^{1-\varepsilon} \right.$$

$$\left. \times \left\| R_{\leqslant M_{k+q-1}}^{(k+q-1)} \tilde{B}_{N,\mu_m(k+q),k+q} \left(f_{KIP}^{(k,q)} \right) \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \right]$$

where the sum is over all $M_k, \ldots, M_{k+q-1}$ dyadic such that $M_{k+q-1} \geqslant \cdots \geqslant M_k \geqslant M_{k-1}$.

Hence

$$\left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} KIP^{k,p} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant \sum_{q=0}^p (C_\varepsilon T^{\frac{1}{2}})^q \sum_{M_{k+q-1} \geqslant \cdots \geqslant M_k \geqslant M_{k-1}} \left[\left(\frac{M_{k-1}}{M_{k+q-1}} \right)^{1-\varepsilon} \right.$$

$$\left. \times \left\| R_{\leqslant M_{k+q-1}}^{(k+q-1)} \tilde{B}_{N,\mu_m(k+q),k+q} \left(f_{KIP}^{(k,q)} \right) \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \right]$$

We then insert a smooth cut-off $\theta(t)$ with $\theta(t) = 1$ for $t \in [-T, T]$ and $\theta(t) = 0$ for $t \in [-2T, 2T]^c$ into the above estimate to get

$$\left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} KIP^{k,p} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant \sum_{q=0}^{p} (C_\varepsilon T^{\frac{1}{2}})^q \sum_{M_{k+q-1} \geqslant \cdots \geqslant M_k \geqslant M_{k-1}} \left[\left(\frac{M_{k-1}}{M_{k+q-1}} \right)^{1-\varepsilon} \right.$$

$$\left. \times \left\| R_{\leqslant M_{k+q-1}}^{(k+q-1)} \tilde{B}_{N,\mu_m(k+q),k+q} \left(\theta(t_{k+q}) \tilde{f}_{KIP}^{(k,q)} \right) \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \right]$$

with

$$(4.14) \qquad \tilde{f}_{KIP}^{(k,q)} = \int_0^{t_{k+q}} U^{(k+q)}(t_{k+q} - t_{k+q+1}) \theta(t_{k+q+1}) \tilde{B}_{N,many}^{(k+q+1)} \alpha_N^{(k+q+1)} (t_{k+q+1}) \, dt_{k+q+1},$$

where the sum is over all $M_k, \ldots, M_{k+q-1}$ dyadic such that $M_{k+q-1} \geqslant \cdots \geqslant M_k \geqslant M_{k-1}$.

4.1.2. *Step II.* With Lemma 5.3, the X_b space version of Lemma 5.2, we turn Step I into

$$\left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} KIP^{k,p} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leqslant \sum_{q=0}^{p} (C_\varepsilon T^{\frac{1}{2}})^{q+1} \sum_{M_{k+q} \geqslant M_{k+q-1} \geqslant \cdots \geqslant M_k \geqslant M_{k-1}} \left[\left(\frac{M_{k-1}}{M_{k+q}} \right)^{1-\varepsilon} \right.$$

$$\left. \times \left\| R_{\leqslant M_{k+q}}^{(k+q)} \left(\theta(t_{k+q}) \tilde{f}_{KIP}^{(k,q)} \right) \right\|_{X_{\frac{1}{2}+}^{(k+q)}} \right].$$

Use Lemma 5.1 gives us

$$\leqslant \sum_{q=0}^{p} (C_\varepsilon T^{\frac{1}{2}})^{q+1} \sum_{M_{k+q} \geqslant M_{k+q-1} \geqslant \cdots \geqslant M_k \geqslant M_{k-1}} \left[\left(\frac{M_{k-1}}{M_{k+q}} \right)^{1-\varepsilon} \right.$$

$$\left. \times \left\| R_{\leqslant M_{k+q}}^{(k+q)} \left(\theta(t_{k+q+1}) \tilde{B}_{N,many}^{(k+q+1)} \alpha_N^{(k+q+1)} (t_{k+q+1}) \right) \right\|_{X_{-\frac{1}{2}+}^{(k+q)}} \right].$$

Carry out the sum in $M_k \leqslant \cdots \leqslant M_{k+q-1}$ with the help of Lemma 4.3:

$$\leqslant \sum_{q=0}^{p} (C_\varepsilon T^{\frac{1}{2}})^{q+1} \sum_{M_{k+q} \geqslant M_{k-1}} \left[\left(\frac{M_{k-1}}{M_{k+q}} \right)^{1-\varepsilon} \left(\frac{(\log_2 \frac{M_{k+q}}{M_{k-1}} + q)^q}{q!} \right) \right.$$

$$\left. \times \left\| R_{\leqslant M_{k+q}}^{(k+q)} \left(\theta(t_{k+q+1}) \tilde{B}_{N,many}^{(k+q+1)} \alpha_N^{(k+q+1)} (t_{k+q+1}) \right) \right\|_{X_{-\frac{1}{2}+}^{(k+q)}} \right].$$

Take a $T^{j/4}$ from the front to apply Lemma 4.4:

$$\leqslant (C_\varepsilon T^{\frac{1}{2}}) \sum_{q=0}^{p} \left\{ (C_\varepsilon T^{\frac{1}{4}})^q \sum_{M_{k+q} \geqslant M_{k-1}} \left[\left(\frac{M_{k-1}^{1-2\varepsilon}}{M_{k+q}^{1-2\varepsilon}} \right) \right. \right.$$

$$\left. \left. \times \left\| R_{\leqslant M_{k+q}}^{(k+q)} \left(\theta(t_{k+q+1}) \tilde{B}_{N,many}^{(k+q+1)} \alpha_N^{(k+q+1)} (t_{k+q+1}) \right) \right\|_{X_{-\frac{1}{2}+}^{(k+q)}} \right] \right\}.$$

where the sum is over dyadic M_{k+q} such that $M_{k+q} \geqslant M_{k-1}$.

Lemma 4.3 ([21, Lemma 3.1]).

$$\left(\sum_{M_{k-1} \leq M_k \leq \cdots \leq M_{k+j-1} \leq M_{k+j}} 1 \right) \leq \frac{(\log_2 \frac{M_{k+j}}{M_{k-1}} + j)^j}{j!},$$

where the sum is in $M_k \leq \cdots \leq M_{k+j-1}$ over dyads, such that $M_{k-1} \leq M_k \leq \cdots \leq M_{k+j-1} \leq M_{k+j}$.

Lemma 4.4 ([21, Lemma 3.2]). *For each $\alpha > 0$ (possibly large) and each $\epsilon > 0$ (arbitrarily small), there exists $t > 0$ (independent of M) sufficiently small such that*

$$\forall\, j \geq 1,\ \forall\, M\,, \quad \text{we have} \quad \frac{t^j (\alpha \log M + j)^j}{j!} \leq M^\epsilon$$

4.1.3. *Step III.* Recall the ending result of Step II,

$$\left\| R^{(k-1)}_{\leqslant M_{k-1}} B_{N,1,k} KIP^{k,p} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$

$$\leqslant\ (C_\varepsilon T^{\frac{1}{2}}) \sum_{q=0}^{p} \left\{ (C_\varepsilon T^{\frac{1}{4}})^q \sum_{M_{k+q} \geqslant M_{k-1}} \left[\left(\frac{M_{k-1}^{1-2\varepsilon}}{M_{k+q}^{1-2\varepsilon}} \right) \right. \right.$$

$$\times \left. \left. \left\| R^{(k+q)}_{\leqslant M_{k+q}} \left(\theta(t_{k+q+1}) \tilde{B}^{(k+q+1)}_{N,many} \alpha^{(k+q+1)}_N \left(t_{k+q+1} \right) \right) \right\|_{X^{(k+q)}_{-\frac{1}{2}+}} \right] \right\}.$$

Use Corollary 4.5,

$$\leqslant\ (C_\varepsilon T^{\frac{1}{2}}) \sum_{q=0}^{p} \left\{ (C_\varepsilon T^{\frac{1}{4}})^q \sum_{M_{k+q} \geqslant M_{k-1}} \left[\left(\frac{M_{k-1}^{1-2\varepsilon}}{M_{k+q}^{1-2\varepsilon}} \right) \right. \right.$$

$$\times C^{k+q+1} N^{-\frac{1}{2}+} \min(N^\beta, M_{k+q}) \frac{1}{\sqrt{N}} \left. \left. \left\| \theta(t_{k+q+1}) S_1 S^{(k+q+1)} \alpha^{(k+q+1)}_N \right\|_{L^2_{t_{k+q+1}} L^2_{\mathbf{x}} L^2_{\mathbf{x}'}} \right] \right\},$$

because there are $(k+q)$ terms inside $\tilde{B}^{(k+q+1)}_{N,many}$. Rearranging terms

$$\leqslant\ C^k (C_\varepsilon T^{\frac{1}{2}}) \sum_{q=0}^{p} \left\{ (CT^{\frac{1}{4}})^q \frac{1}{\sqrt{N}} \left\| \theta(t_{k+q+1}) S_1 S^{(k+q+1)} \alpha^{(k+q+1)}_N \right\|_{L^2_{t_{k+q+1}} L^2_{\mathbf{x}} L^2_{\mathbf{x}'}} \right.$$

$$M_{k-1}^{1-2\varepsilon} N^{-\frac{1}{2}+} \sum_{M_{k+q} \geqslant M_{k-1}} \min(M_{k+q}^{-1+2\varepsilon} N^\beta, M_{k+q}^{2\varepsilon}) \Big\},$$

We carry out the sum in M_{k+q} by dividing into $M_{k+q} \leqslant N^\beta$ (for which $\min(M_{k+q}^{-1+2\varepsilon} N^\beta, M_{k+q}^{2\varepsilon}) = M_{k+q}^{2\epsilon})$ and $M_{k+q} \geqslant N^\beta$ (for which $\min(M_{k+q}^{-1+2\varepsilon} N^\beta, M_{k+q}^{2\varepsilon}) = M_{k+q}^{-1+2\varepsilon} N^\beta$). This yields

$$(4.15) \quad \sum_{M_{k+q} \geqslant M_{k-1}} \min(M_{k+q}^{-1+2\varepsilon} N^\beta, M_{k+q}^{2\varepsilon}) \lesssim \sum_{N^\beta \geqslant M_{k+q} \geqslant M_{k-1}} (\ldots) + \sum_{M_{k+q} \geqslant M_{k-1}, M_{k+q} \geqslant N^\beta} (\ldots)$$

$$\lesssim \sum_{N^\beta \geqslant M_{k+q} \geqslant M_{k-1}} M_{k+q}^{2\epsilon} + \sum_{M_{k+q} \geqslant N^\beta} M_{k+q}^{-1+2\varepsilon} N^\beta$$

$$\lesssim N^{2\epsilon}.$$

Remark 2. *The above is exactly what we meant by writing "gains one derivative via Littlewood-Paley" in §1.1.*

So we have reached

$$\left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} KIP^{k,p} \right\|_{L_T^1 L_{\mathbf{x}, \mathbf{x}'}^2}$$

$$\leqslant C^k (C_\varepsilon T^{\frac{1}{2}}) \sum_{q=0}^{p} \left\{ (CT^{\frac{1}{4}})^q \frac{1}{\sqrt{N}} \left\| \theta(t_{k+q+1}) S_1 S^{(k+q+1)} \alpha_N^{(k+q+1)} \right\|_{L_{t_{k+q+1}}^2 L_{\mathbf{x}}^2 L_{\mathbf{x}'}^2} \right.$$

$$\left. M_{k-1}^{1-2\varepsilon} N^{-\frac{1}{2}+2\varepsilon} \right\}.$$

Via Condition (1.19) (the energy estimate), it becomes

$$\leqslant C^k (C_\varepsilon T^{\frac{1}{2}}) \sum_{q=0}^{p} (CT^{\frac{1}{4}})^q C_0^{k+q+1} M_{k-1}^{1-2\varepsilon} N^{-\frac{1}{2}+2\varepsilon}$$

$$\leqslant C^k (C_\varepsilon T^{\frac{1}{2}}) \sum_{q=0}^{\infty} (CT^{\frac{1}{4}})^q C_0^{q+1} M_{k-1}^{1-2\varepsilon} N^{-\frac{1}{2}+2\varepsilon}.$$

We can then choose a T independent of M_{k-1}, k, p and N such that the infinite series converges. We then have

$$\left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} KIP^{k,p} \right\|_{L_T^1 L_{\mathbf{x}, \mathbf{x}'}^2} \leqslant C^{k-1} M_{k-1}^{1-2\varepsilon} N^{-\frac{1}{2}+2\varepsilon}$$

for some C larger than C_0. Therefore, on the one hand, there is a C independent of M_{k-1}, k, p, and N s.t. given a M_{k-1}, there is $N_0(M_{k-1})$ which makes

$$\left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} KIP^{k,p} \right\|_{L_T^1 L_{\mathbf{x}, \mathbf{x}'}^2} \leqslant C^{k-1}, \text{ for all } N \geqslant N_0,$$

on the other hand,

$$\left\| R_{\leqslant M_{k-1}}^{(k-1)} B_{N,1,k} KIP^{k,p} \right\|_{L_T^1 L_{\mathbf{x}, \mathbf{x}'}^2} \to 0 \text{ as } N \to \infty$$

which matches Theorem 1.1 as well. Whence we have finished the proof of estimate (4.5).

Corollary 4.5.

$$\left\| R^{(k+q)}_{\leq M_{k+q}} \tilde{B}^{(k+q+1)}_{N,\text{many}} \alpha^{(k+q+1)}_N \right\|_{X^{(k+q)}_{-\frac{1}{2}+}}$$

(4.16)

$$\lesssim C^{k+q} \left(N^{-\frac{1}{2}} \| S_1 S^{(k+q+1)} \alpha^{(k+q+1)}_N \|_{L^2_t L^2_{\mathbf{xx}'}} \right) \begin{cases} M_{k+q} N^{-\frac{1}{2}+} \\ N^{\beta - \frac{1}{2}+} \end{cases}$$

Proof. Recall (2.10), which gives the expansion

(4.17)
$$\tilde{B}^{(k+q)}_{N,\text{many}} = \sum_{\ell=1}^{k+q} \tilde{B}_{N,\text{many},\ell,k+q+1}$$

where $\tilde{B}_{N,\text{many},\ell,k+q+1}$ is defined by (2.11) and itself decomposed in Lemma 3.3 into a sum of at most 8^{k+q} terms of the form

(4.18)
$$\beta^{(k+q)}_N = \frac{N-k}{N} \int_{\mathbb{R}^3} \tilde{V}_N(x_l - x_{k+q+1}) N^{\beta-1} w_0(N^\beta(x_\sigma - x_{k+q+1}))$$

$$A_\sigma \alpha^{(k+q+1)}_N (\mathbf{x}_{k+q}, x_{k+q+1}; \mathbf{x}'_{k+q}, x_{k+q+1}) \, dx_{k+q+1}.$$

Here, $x_\sigma \in \{x_1, \ldots, x_{k+q+1}, x'_1, \ldots, x'_{k+q+1}\} \setminus \{x_\ell\}$ and

$$A_\sigma = \prod_{\substack{1 \leq j \leq k+q \\ j \neq \ell, \, j \neq \sigma}} Z_j Z'_j$$

where Z_j is either 1 or $N^{\beta-1} w_0(N^\beta(x_j - x_{k+q+1}))$, and likewise Z'_j is either 1 or $N^{\beta-1} w_0(N^\beta(x'_j - x_{k+q+1}))$. Since there are $(k+q)$ terms in (4.17) and $\leq 8^{k+q}$ terms of the type $\beta^{(k+q)}_N$ in (4.18), we multiply by a factor C^{k+q}. For each individual term $\beta^{(k+q)}_N$, the derivatives ∇_{x_j} for $1 \leq j \leq k+q$, $j \neq \ell$, $j \neq \sigma$ can either land on $Z_j Z'_j$ or $\alpha^{(k+q+1)}_N$, giving 2^{k+q-1} terms. Each possibility is accommodated by a suitable variant of Proposition 5.13. Of course, we actually need to modify (5.25) so that it has a $(k+q+1)$-component density (as opposed to 4) and multiple factors of the type $f_N(x_1 - x_4)$ in (5.25), but these modifications are straightforward and amount to bookkeeping. The remaining coordinates act as "passive variables" and are placed in L^2 on the inside of the estimates, and otherwise do not play any role. $\qquad \square$

4.2. **Estimate for the Potential Part.** Repeating Steps I and II in the treatment of the k-body interaction part, we have

$$\left\| R^{(k-1)}_{\leqslant M_{k-1}} B_{N,1,k} PP^{k,p} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$

$$\leqslant (C_\varepsilon T^{\frac{1}{2}}) \sum_{q=0}^p \left\{ (C_\varepsilon T^{\frac{1}{4}})^q \sum_{M_{k+q} \geqslant M_{k-1}} \left[\left(\frac{M^{1-2\varepsilon}_{k-1}}{M^{1-2\varepsilon}_{k+q}} \right) \right.\right.$$

$$\times \left.\left. \left\| R^{(k+q)}_{\leqslant M_{k+q}} \left(\theta(t_{k+q+1}) \tilde{V}^{(k+q)}_N \alpha^{(k+q)}_N (t_{k+q+1}) \right) \right\|_{X^{(k+q)}_{-\frac{1}{2}+}} \right] \right\}.$$

Recall

$$\tilde{V}^{(k)}_N \alpha^{(k)}_N = (A^{(k)}_N - A^{(k)'}_N) \alpha^{(k)}_N + (E^{(k)}_N - E^{(k)'}_N) \alpha^{(k)}_N.$$

From here on out, we will call $(A_N^{(k)} - A_N^{(k)'})\alpha_N^{(k)}$ the three-body potential term and $(E_N^{(k)} - E_N^{(k)'})\alpha_N^{(k)}$ the two-body error term.

By Step III in the estimate of the k-body interaction term, it suffices to prove the following two corollaries.

Corollary 4.6. *Recall*

$$A_{N,i,j,\ell}^{(k+q)}\alpha^{(k+q)} = \frac{\nabla_{x_\ell}G_{N,\ell,i} \cdot \nabla_{x_\ell}G_{N,\ell,j}}{G_{N,\ell,i}\, G_{N,\ell,j}}\alpha^{(k+q)},$$

as defined in (3.4). *Then*

$$\|R_{\leq M_{k+q}}^{(k+q)}A_{N,i,j,\ell}\alpha_N^{(k+q)}\|_{X_{-\frac{1}{2}+}^{(k+q)}} \lesssim \|S^{(k+q)}\alpha_N^{(k+q)}\|_{L_t^2 L_{\mathbf{xx'}}^2} \begin{cases} N^{3\beta-2} \\ M_{k+q}^3 N^{-2+} \end{cases}$$

Corollary 4.7. *Recall*

$$E_{N,j,l}^{(k+q)}\alpha^{(k+q)} = \frac{\nabla_{x_\ell}G_{N,j,\ell}}{G_{N,j,\ell}} \cdot \nabla_{x_\ell}\alpha^{(k+q)}$$

as defined in (3.5), we have

(4.19)
$$\left\|R_{\leq M_{k+q}}^{(k+q)}\left(E_{N,j,l}^{(k)}\alpha^{(k)}\left(t_{k+1}\right)\right)\right\|_{X_{-\frac{1}{2}+}^{(k+q)}}$$
$$\lesssim \left(N^{-\frac{1}{2}}\|S_1 S^{(k+q)}\alpha\|_{L_t^2 L_{\mathbf{x}_{k+q}\mathbf{x}'_{k+q}}^2} + \|S^{(k)}\alpha\|_{L_t^2 L_{\mathbf{x}_{k+q}\mathbf{x}'_{k+q}}^2}\right)\begin{cases} M_{k+q}N^{\frac{\beta}{2}-\frac{3}{4}} \\ N^{\frac{3\beta}{2}-\frac{3}{4}} \end{cases}$$

where, for convenience, we have assumed that $\beta > \frac{1}{2}$.

Then one merely needs to estimate the following two sums:

$$N^{-2+}\sum_{M_{k+q}\geqslant M_{k-1}}\min\left(M_{k+q}^{-1+2\varepsilon}N^{3\beta}, M_{k+q}^{2+2\varepsilon}\right),$$

and

$$N^{\frac{1}{2}\beta-\frac{3}{4}+}\sum_{M_{k+q}\geqslant M_{k-1}}\min\left(M_{k+q}^{-1+2\varepsilon}N^{\beta}, M_{k+q}^{2\varepsilon}\right).$$

In fact, separate the above sums at $M_{k+q}\geqslant N^{\beta}$ and $M_{k+q}\leqslant N^{\beta}$, then use the same method as in estimate (4.15), we get to

$$N^{-2+}\sum_{M_{k+q}\geqslant M_{k-1}}\min\left(M_{k+q}^{-1+2\varepsilon}N^{3\beta}, M_{k+q}^{2+2\varepsilon}\right) \lesssim N^{-2+}N^{2\beta+2\varepsilon}$$

$$N^{\frac{1}{2}\beta-\frac{3}{4}+}\sum_{M_{k+q}\geqslant M_{k-1}}\min\left(M_{k+q}^{-1+2\varepsilon}N^{\beta}, M_{k+q}^{2\varepsilon}\right) \lesssim N^{\frac{1}{2}\beta-\frac{3}{4}+}N^{2\varepsilon}$$

which is enough to conclude the estimates of the potential part for $\beta \in (0,1)$.

Remark 3. *We remark that the estimate for the three-body interaction term is the only place in this paper which requires $\beta < 1$.*

Proof of Corollary 4.6. Since ∇_{x_μ} and $\nabla_{x'_\mu}$ move directly onto $\alpha_N^{(k+q)}$, for $\mu \in \{1, \dots, k + q\} \backslash \{i, j, \ell\}$, it suffices to use the obvious extension of Proposition 5.11 where $\{1, 2, 3\}$ is replaced by $\{\ell, i, j\}$, $\alpha^{(3)}$ is replaced by $\alpha^{(k+q)}$, and $X_{-\frac{1}{2}+}^{(3)}$ is replaced by $X_{-\frac{1}{2}+}^{(k+q)}$. Note that

$$A_{Nij\ell} = N^{-2\beta-2} U_N(x_\ell - x_i) U_N(x_\ell - x_j)$$

where

$$U(x) = \frac{(\nabla w_0)(x)}{1 - N^{\beta-1} w_0(x)}$$

Note that

$$\nabla U(x) = \frac{\nabla^2 w_0}{(1 - w_0(x))^2} + N^{\beta-1} \left(\frac{1}{1 - w_0(x)} \right)^2$$

and

$$|U(x)| \lesssim \langle x \rangle^{-2}, \qquad |\nabla U(x)| \lesssim \langle x \rangle^{-3}, \qquad |\nabla^2 U(x)| \lesssim \langle x \rangle^{-4}$$

uniformly in N. Hence U, ∇U, and $\nabla^2 U$ all belong to L^p for $p > \frac{3}{2}$ (uniformly in N). □

Proof of Corollary 4.7. Note that

$$\frac{\nabla_{x_\ell} G_{N,j,\ell}}{G_{N,j,\ell}} = N^{-\beta-1} U_N(x_j - x_\ell)$$

where

$$U(x) = \frac{\nabla w_0(x)}{1 - N^{\beta-1} w_0(x)}$$

We then appeal to the straightforward generalization of Proposition 5.7 to $(k + q)$-level density, noting that $|U(x)| \lesssim \langle x \rangle^{-2}$, $|\nabla U(x)| \lesssim \langle x \rangle^{-3}$, and $|\nabla^2 U(x)| \lesssim \langle x \rangle^{-4}$, uniformly in N, so $C_U < \infty$ and independent of N. □

5. COLLAPSING AND STRICHARTZ ESTIMATES

Define the norm[7]

$$\|\alpha^{(k)}\|_{X_b^{(k)}} = \left(\int \langle \tau + |\boldsymbol{\xi}_k|^2 - |\boldsymbol{\xi}'_k|^2 \rangle^{2b} \left| \hat{\alpha}^{(k)}(\tau, \boldsymbol{\xi}_k, \boldsymbol{\xi}'_k) \right|^2 d\tau \, d\boldsymbol{\xi}_k \, d\boldsymbol{\xi}'_k \right)^{1/2}$$

We will use the case $b = \frac{1}{2}+$ of the following lemma.

Lemma 5.1 ([21, Lemma 4.1]). *Let* $\frac{1}{2} < b < 1$ *and* $\theta(t)$ *be a smooth cutoff. Then*

$$(5.1) \qquad \left\| \theta(t) \int_0^t U^{(k)}(t - s) \beta^{(k)}(s) \, ds \right\|_{X_b^{(k)}} \lesssim \|\beta^{(k)}\|_{X_{b-1}^{(k)}}$$

Lemma 5.2 ([21, Lemma 4.4]). *For each* $\varepsilon > 0$, *there is a* C_ε *independent of* M_k, j, k, *and* N *such that*

$$\|R_{\leqslant M_k}^{(k)} \tilde{B}_{N,j,k+1} U^{(k+1)}(t) f^{(k+1)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C_\varepsilon \left\| \tilde{V} \right\|_{L^1} \sum_{M_{k+1} \geqslant M_k} \left(\frac{M_k}{M_{k+1}} \right)^{1-\varepsilon} \left\| R_{\leqslant M_{k+1}}^{(k+1)} f^{(k+1)} \right\|_{L_{\mathbf{x},\mathbf{x}'}^2}$$

where the sum on the right is in M_{k+1}, *over dyads such that* $M_{k+1} \geqslant M_k$.

---

[7]To be precise, this X_b should be written as $X_{0,b}$ in the usual notation for the $X_{s,b}$ spaces. Since we are not using the s in $X_{s,b}$, we write it as X_b.

Lemma 5.3 ([21, Lemma 4.5]). *For each $\varepsilon > 0$, there is a C_ε independent of M_k, j, k, and N such that*

$$\|R^{(k)}_{\leqslant M_k} \tilde{B}_{N,j,k+1} \alpha^{(k+1)}\|_{L^2_t L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C_\varepsilon \sum_{M_{k+1} \geqslant M_k} \left(\frac{M_k}{M_{k+1}}\right)^{1-\varepsilon} \left\|R^{(k+1)}_{\leqslant M_{k+1}} \alpha^{(k+1)}\right\|_{X^{(k)}_{\frac{1}{2}+}}.$$

where the sum on the right is in M_{k+1}, over dyads such that $M_{k+1} \geqslant M_k$.

The 3D endpoint Strichartz estimate directly yields the following multiparticle estimate:

$$\|\beta^{(k)}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim \|\beta^{(k)}\|_{L^2_t L^{\frac{6}{5}+}_{x_1} L^2_c}$$

where c stands for the remaining spatial coordinates $(x_2, \ldots, x_k, x'_1, \ldots, x'_k)$. However, when $\beta^{(k)} = V(x_1 - x_2)\gamma^{(k)}$, this estimate does not allow us to put V in $L^{\frac{6}{5}+}$ since the $L^2_{x_2}$ norm comes before the $L^{\frac{6}{5}+}_{x_1}$ norm. In order to effectively put the $L^2_{x_2}$ norm *after* the $L^{\frac{6}{5}+}_{x_1}$ norm, we need to translate coordinates before applying the Strichartz estimate. This maneuver was introduced in our earlier paper [20, Lemma 4.6]. We restate the relevant estimate in the following lemma.

Since we will need to deal with Fourier transforms in only selected coordinates, we introduce the following notation: $\mathcal{F}_0$ denotes the Fourier transform in t, $\mathcal{F}_j$ denotes the Fourier transform in x_j, and $\mathcal{F}_{j'}$ denotes Fourier transform in x'_j. Fourier transforms in multiple coordinates will be denoted as combined subscripts – for example, $\mathcal{F}_{01'} = \mathcal{F}_0 \mathcal{F}_{1'}$ denotes the Fourier transform in t and x'_1.

Lemma 5.4 (3D endpoint Strichartz in transformed coordinates). *Let*

$$T_1 f(x_1, x_2) = f(x_1 + x_2, x_2)$$

$$T_2 f(x_1, x_2) = f(x_1, x_2 + x_1)$$

Then

$$(5.2) \qquad \|\beta^{(k)}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim \begin{cases} \|(\mathcal{F}_2 T_1 \beta^{(k)})(t, x_1, \xi_2)\|_{L^2_t L^2_{\xi_2} L^{\frac{6}{5}+}_{x_1} L^2_c} \\[2mm] \|(\mathcal{F}_1 T_2 \beta^{(k)})(t, \xi_1, x_2)\|_{L^2_t L^2_{\xi_1} L^{\frac{6}{5}+}_{x_2} L^2_c} \end{cases}$$

where in each case c stands for "complementary coordinates", specifically coordinates $(x_3, \ldots, x_k, x'_1, \ldots, x'_k)$.

Lemma 5.5 (Hölder and Sobolev). *If*

$$(5.3) \qquad \beta^{(k)}(t, x_1, x_2) = V(x_1 - x_2)\gamma^{(k)}(t, x_1, x_2)$$

then

$$(5.4) \qquad \|(\mathcal{F}_2 T_1 \beta^{(k)})(t, x_1, \xi_2)\|_{L^2_t L^2_{\xi_2} L^{\frac{6}{5}+}_{x_1} L^2_c} \lesssim \begin{cases} \|V\|_{L^{\frac{6}{5}+}} \|\langle\nabla_{x_1}\rangle^{\frac{3}{2}} \gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \\[2mm] \|V\|_{L^{\frac{3}{2}+}} \|\nabla_{x_1} \gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \\[2mm] \|V\|_{L^{3+}} \|\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \end{cases}$$

$$(5.5) \qquad \|(\mathcal{F}_1 T_2 \beta^{(k)})(t, \xi_1, x_2)\|_{L_t^2 L_{\xi_1}^2 L_{x_2}^{\frac{6}{5}+} L_c^2} \lesssim \begin{cases} \|V\|_{L^{\frac{6}{5}+}} \|\langle \nabla_{x_2} \rangle^{\frac{3}{2}} \gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx}'}^2} \\ \|V\|_{L^{\frac{3}{2}+}} \|\nabla_{x_2} \gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx}'}^2} \\ \|V\|_{L^{3+}} \|\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx}'}^2} \end{cases}$$

Proof. Consider (5.4). By (5.3),

$$(T_1 \beta^{(k)})(t, x_1, x_2) = V(x_1)(T_1 \gamma^{(k)})(t, x_1, x_2)$$

and hence, applying $\mathcal{F}_2$, we obtain

$$(\mathcal{F}_2 T_1 \beta^{(k)})(t, x_1, \xi_2) = V(x_1)(\mathcal{F}_2 T_1 \gamma^{(k)})(t, x_1, \xi_2)$$

Applying Hölder,

$$\|(\mathcal{F}_2 T_1 \beta^{(k)})(t, x_1, \xi_2)\|_{L_{x_1}^{\frac{6}{5}+} L_c^2} \leq \|V\|_{L^{\frac{6}{5}+}} \|(\mathcal{F}_2 T_1 \gamma^{(k)})(t, x_1, \xi_2)\|_{L_{x_1}^{\infty-} L_c^2}$$

By Sobolev,

$$\|(\mathcal{F}_2 T_1 \beta^{(k)})(t, x_1, \xi_2)\|_{L_{x_1}^{\frac{6}{5}+} L_c^2} \leq \|V\|_{L^{\frac{6}{5}+}} \|\langle \nabla_{x_1} \rangle^{\frac{3}{2}} (\mathcal{F}_2 T_1 \gamma^{(k)})(t, x_1, \xi_2)\|_{L_{x_1}^2 L_c^2}$$

$$= \|V\|_{L^{\frac{6}{5}+}} \|\mathcal{F}_2 \langle \nabla_{x_1} \rangle^{\frac{3}{2}} (T_1 \gamma^{(k)})(t, x_1, \xi_2)\|_{L_{x_1}^2 L_c^2}$$

Applying the $L_{\xi_2}^2$ norm and Plancherel in x_2,

$$\|(\mathcal{F}_2 T_1 \beta^{(k)})(t, x_1, \xi_2)\|_{L_{\xi_2}^2 L_{x_1}^{\frac{6}{5}+} L_c^2} = \|V\|_{L^{\frac{6}{5}+}} \|\langle \nabla_{x_1} \rangle^{\frac{3}{2}} (T_1 \gamma^{(k)})(t, x_1, x_2)\|_{L_{\mathbf{xx}'}^2}$$

Reviewing the definition of T_1, we see that

$$= \|V\|_{L^{\frac{6}{5}+}} \|\langle \nabla_{x_1} \rangle^{\frac{3}{2}} \gamma^{(k)}(t, x_1, x_2)\|_{L_{\mathbf{xx}'}^2}.$$

The other cases are similar. $\qquad \square$

Using frequency localization, we can share derivatives between two coordinates, as in the following corollary.

Corollary 5.6. *If $\gamma^{(k)}$ is symmetric and*

$$\beta^{(k)}(t, x_1, x_2) = V(x_1 - x_2) \gamma^{(k)}(t, x_1, x_2)$$

then

$$(5.6) \qquad \|\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \lesssim \begin{cases} \|V\|_{L^{\frac{6}{5}+}} \|\langle \nabla_{x_1} \rangle^{\frac{3}{4}} \langle \nabla_{x_2} \rangle^{\frac{3}{4}} \gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx}'}^2} \\ \|V\|_{L^{\frac{3}{2}+}} \|\langle \nabla_{x_i} \rangle \gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx}'}^2} \quad with \ i = 1, 2 \\ \|V\|_{L^{3+}} \|\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx}'}^2} \end{cases}$$

Proof. We need only to prove the first inequality of (5.6). The other two are directly from Lemma 5.5 and the fact that $\gamma^{(k)}$ is symmetric i.e. $\|\langle \nabla_{x_1} \rangle \gamma^{(k)}\| = \|\langle \nabla_{x_2} \rangle \gamma^{(k)}\|$.

Split $\gamma^{(k)}$ according to the relative magnitude of the ξ_1 and ξ_2 frequencies:

$$\gamma^{(k)} = P_{|\xi_1| \leq |\xi_2|} \gamma^{(k)} + P_{|\xi_2| \leq |\xi_1|} \gamma^{(k)}$$

and define
$$\beta_{1\leq2}^{(k)} \stackrel{\text{def}}{=} V(x_1 - x_2)P_{|\xi_1|\leq|\xi_2|}\gamma^{(k)}$$
$$\beta_{2\leq1}^{(k)} \stackrel{\text{def}}{=} V(x_1 - x_2)P_{|\xi_2|\leq|\xi_1|}\gamma^{(k)}$$

so that
$$\beta^{(k)} = \beta_{1\leq2}^{(k)} + \beta_{2\leq1}^{(k)}$$

For the $\beta_{1\leq2}^{(k)}$ piece, use the first estimate in (5.2) together with the first estimate of (5.4) to obtain

$$\|\beta_{1\leq2}^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \lesssim \|V\|_{L^{\frac{6}{5}+}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{2}}P_{|\xi_1|\leq|\xi_2|}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2}$$

$$\lesssim \|V\|_{L^{\frac{6}{5}+}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{4}}\langle\nabla_{x_2}\rangle^{\frac{3}{4}}P_{|\xi_1|\leq|\xi_2|}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2}$$

$$\lesssim \|V\|_{L^{\frac{6}{5}+}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{4}}\langle\nabla_{x_2}\rangle^{\frac{3}{4}}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2}$$

where, in the middle line, we used the frequency restriction to $|\xi_1| \leq |\xi_2|$.

For the $\beta_{2\leq1}^{(k)}$ piece, use the second estimate in (5.2) together with the first estimate of (5.5), and proceed in an analogous fashion to obtain

$$\|\beta_{2\leq1}^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \lesssim \|V\|_{L^{\frac{6}{5}+}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{4}}\langle\nabla_{x_2}\rangle^{\frac{3}{4}}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2}$$

$\square$

Using Corollary 5.6, we can prove the following proposition which will be for the first order term in the PP estimate.

Proposition 5.7. *For any $U(x)$, let $U_N(x) = N^{3\beta}U(N^\beta x)$. Then*

$$\|R_{\leq M_2}^{(2)}\left(U_N(x_1 - x_2)\nabla_{x_2}\alpha^{(2)}\right)\|_{X_{-\frac{1}{2}+}^{(2)}}$$

(5.7)
$$\lesssim C_U\left(N^{-\frac{1}{2}}\|S_1 S^{(2)}\alpha\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2} + \|S^{(2)}\alpha\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}\right)\begin{cases} M_2(N^{\frac{3\beta}{2}+\frac{1}{4}+} + N^{\beta+\frac{1}{2}+}) \\ (N^{\frac{5\beta}{2}+\frac{1}{4}+} + N^{2\beta+\frac{1}{2}+}) \end{cases}$$

where

$$C_U = \|\nabla^2 U\|_{L^{\frac{6}{5}+}} + \|\nabla U\|_{L^{\frac{6}{5}+}\cap L^{\frac{3}{2}+}} + \|U\|_{L^{\frac{3}{2}+}\cap L^{3+}}$$

Proof. We begin by proving the first estimate of (5.7). Let

$$\beta^{(2)} = R_{\leq M_2}^{(2)}\left(U_N(x_1 - x_2)\nabla_{x_2}\alpha^{(2)}\right) = P_{\leq M_2}^{(2)}\nabla_{x_1}\nabla_{x_2}\left(U_N(x_1 - x_2)\nabla_{x_2}\alpha^{(2)}\right) = \text{A} + \text{B}$$

where A and B are produced by distributing ∇_{x_1} into the product:

$$\text{A} = N^\beta P_{\leq M_2}^{(2)}\nabla_{x_2}\left((\nabla U)_N(x_1 - x_2)\nabla_{x_2}\alpha^{(2)}\right)$$

$$\text{B} = P_{\leq M_2}^{(2)}\nabla_{x_2}\left(U_N(x_1 - x_2)\nabla_{x_1}\nabla_{x_2}\alpha^{(2)}\right)$$

Using $P_{\leq M_2}^{(2)}\nabla_{x_2} \leq M_2$, we have

$$\|\text{A}\|_{X_{-\frac{1}{2}+}} \lesssim M_2 N^\beta\|(\nabla U)_N(x_1 - x_2)\nabla_{x_2}\alpha^{(2)}\|_{X_{-\frac{1}{2}+}}$$

By the first estimate of (5.2) combined with the first estimate of (5.4), we obtain

$$\lesssim M_2 N^\beta \|(\nabla U)_N\|_{L^{\frac{6}{5}+}} \|\langle\nabla_{x_1}\rangle^{\frac{3}{2}} \langle\nabla_{x_2}\rangle\alpha^{(2)}\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}$$

$$= M_2 N^{\frac{3\beta}{2}+\frac{1}{4}+}\|\nabla U\|_{L^{\frac{6}{5}+}}\left(N^{-\frac{1}{4}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{2}}\langle\nabla_{x_2}\rangle\alpha^{(2)}\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}\right)$$

Using $P^{(2)}_{\leq M_2}\nabla_{x_2} \leq M_2$, we have

$$\|\mathrm{B}\|_{X_{-\frac{1}{2}+}} \lesssim M_2 \|U_N(x_1-x_2)\nabla_{x_1}\nabla_{x_2}\alpha^{(2)}\|_{X_{-\frac{1}{2}+}}$$

By the second estimate of (5.6), we obtain

$$\lesssim M_2\|U_N\|_{L^{\frac{3}{2}+}}\|\langle\nabla_{x_1}\rangle^2\langle\nabla_{x_2}\rangle\alpha^{(2)}\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}$$

$$= M_2 N^{\beta+\frac{1}{2}+}\|U\|_{L^{\frac{3}{2}+}}\left(N^{-\frac{1}{2}}\||\nabla_{x_1}|^{\frac{3}{2}}|\nabla_{x_2}|\alpha^{(2)}\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}\right)$$

Now we turn to the second estimate of (5.7). In this case, we distribute both ∇_{x_1} and ∇_{x_2} into the product to obtain 4 terms

$$\beta^{(2)} = R^{(2)}_{\leq M_2}\left(U_N(x_1-x_2)\nabla_{x_2}\alpha^{(2)}\right) = P^{(2)}_{\leq M_2}\nabla_{x_1}\nabla_{x_2}\left(U_N(x_1-x_2)\nabla_{x_2}\alpha^{(2)}\right)$$

$$= \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F}$$

where

$$\mathrm{C} = -N^{2\beta}P^{(2)}_{\leq M_2}(\nabla^2 U)_N(x_1-x_2)\nabla_{x_2}\alpha^{(2)}$$

$$\mathrm{D} = N^\beta P^{(2)}_{\leq M_2}(\nabla U)_N(x_1-x_2)\nabla_{x_1}\nabla_{x_2}\alpha^{(2)}$$

$$\mathrm{E} = -N^\beta P^{(2)}_{\leq M_2}(\nabla U)_N(x_1-x_2)\nabla_{x_2}^2\alpha^{(2)}$$

$$\mathrm{F} = P^{(2)}_{\leq M_2}U_N(x_1-x_2)\nabla_{x_1}\nabla_{x_2}^2\alpha^{(2)}$$

By the first estimate in (5.2) followed by the first estimate in (5.4),

$$\|\mathrm{C}\|_{X_{-\frac{1}{2}+}} \lesssim N^{2\beta}\|(\nabla^2 U)_N(x_1-x_2)\nabla_{x_2}\alpha^{(2)}\|_{X_{-\frac{1}{2}+}}$$

$$\lesssim N^{2\beta}\|(\nabla^2 U)_N\|_{L^{\frac{6}{5}+}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{2}}\nabla_{x_2}\alpha^{(2)}\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}$$

$$= N^{\frac{5\beta}{2}+\frac{1}{4}+}\|\nabla^2 U\|_{L^{\frac{6}{5}+}}\left(N^{-\frac{1}{4}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{2}}\nabla_{x_2}\alpha^{(2)}\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}\right)$$

By the second estimate of (5.6)

$$\|\mathrm{D}\|_{X_{-\frac{1}{2}+}} \lesssim N^\beta\|(\nabla U)_N\|_{L^{\frac{3}{2}+}}\|\langle\nabla_{x_1}\rangle^2\nabla_{x_2}\alpha^{(2)}\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}$$

$$\lesssim N^{2\beta+\frac{1}{2}+}\|\nabla U\|_{L^{\frac{3}{2}+}}\left(N^{-\frac{1}{2}}\|\langle\nabla_{x_1}\rangle^2\nabla_{x_2}\alpha^{(2)}\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}\right)$$

By the second estimate of (5.6). The treatment of E is nearly identical. By the third estimate of (5.6)

$$\|\mathrm{F}\|_{X_{-\frac{1}{2}+}} \lesssim \|U_N\|_{L^{3+}}\|\langle\nabla_{x_1}\rangle^2\nabla_{x_2}\alpha^{(2)}\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}$$

$$\lesssim N^{2\beta+\frac{1}{2}+}\|U\|_{L^{3+}}\left(N^{-\frac{1}{2}}\|\langle\nabla_{x_1}\rangle^2\nabla_{x_2}\alpha^{(2)}\|_{L_t^2 L_{\mathbf{x}_2\mathbf{x}_2'}^2}\right)$$

$\square$

We now provide 6D analogues to the above coordinate translated 3D Strichartz estimate in Lemma 5.4 and the associated Hölder and Sobolev estimates in Lemma 5.5. These 6D estimates are essential to optimally distribute the derivatives in three-body estimates.

Lemma 5.8 (6D endpoint Strichartz in transformed coordinates). [8] *Let*

$$T_{12}f(x_1, x_2, x_3) = f(x_1 + x_3, x_2 + x_3, x_3)$$

$$T_{23}f(x_1, x_2, x_3) = f(x_1, x_2 + x_1, x_3 + x_1)$$

$$T_{13}f(x_1, x_2, x_3) = f(x_1 + x_2, x_2, x_3 + x_2)$$

Then

$$(5.8) \qquad \|\beta^{(k)}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim \begin{cases} \left\|(\mathcal{F}_3 T_{12}\beta^{(k)})(t, x_1, x_2, \xi_3)\right\|_{L^2_t L^2_{\xi_3} L^{\frac{3}{2}+}_{x_1 x_2} L^2_c} \\[2mm] \left\|(\mathcal{F}_2 T_{13}\beta^{(k)})(t, x_1, \xi_2, x_3)\right\|_{L^2_t L^2_{\xi_2} L^{\frac{3}{2}+}_{x_1 x_3} L^2_c} \\[2mm] \left\|(\mathcal{F}_1 T_{23}\beta^{(k)})(t, \xi_1, x_2, x_3)\right\|_{L^2_t L^2_{\xi_1} L^{\frac{3}{2}+}_{x_2 x_3} L^2_c} \end{cases}$$

where in each case c stands for "complementary coordinates", specifically coordinates $(x_4, \ldots, x_k, x'_1, \ldots, x'_k)$.

Proof. We will only prove the first estimate in (5.8). The other two estimates follow in analogy or can be deduced from the first estimate by permuting coordinates (this does not require symmetry of $\beta^{(k)}$).

$$(5.9) \qquad (\mathcal{F}_{123}T_{12}\beta^{(k)})(t, \xi_1, \xi_2, \xi_3) = (\mathcal{F}_{123}\beta^{(k)})(t, \xi_1, \xi_2, \xi_3 - \xi_1 - \xi_2)$$

Also

$$(5.10) \qquad \begin{aligned} &e^{-2it\xi_1 \cdot \xi_3} e^{-2it\xi_2 \cdot \xi_3} (\mathcal{F}_{123}T_{12}\beta^{(k)})(t, \xi_1, \xi_2, \xi_3) \\ &= \mathcal{F}_{12}[(\mathcal{F}_3 T_{12}\beta^{(k)})(t, x_1 - 2t\xi_3, x_2 - 2t\xi_3, \xi_3)](\xi_1, \xi_2) \end{aligned}$$

Now

$$\begin{aligned} &(\mathcal{F}_{0123}\beta^{(k)})(\tau - |\xi_3|^2 + 2\xi_1 \cdot \xi_3 + 2\xi_2 \cdot \xi_3, \xi_1, \xi_2, \xi_3 - \xi_1 - \xi_2) \\ &= (\mathcal{F}_{0123}T_{12}\beta^{(k)})(\tau - |\xi_3|^2 + 2\xi_1 \cdot \xi_3 + 2\xi_2 \cdot \xi_3, \xi_1, \xi_2, \xi_3) && \text{by (5.9)} \\ &= \mathcal{F}_0[e^{it|\xi_3|^2} e^{-2it\xi_1 \cdot \xi_3} e^{-2it\xi_2 \cdot \xi_3} (\mathcal{F}_{123}T_{12}\beta^{(k)})(t, \xi_1, \xi_2, \xi_3)](\tau) \\ &= \mathcal{F}_0[e^{it|\xi_3|^2} \mathcal{F}_{12}[(\mathcal{F}_3 T_{12}\beta^{(k)})(t, x_1 - 2t\xi_3, x_2 - 2t\xi_3, \xi_3)](\xi_1, \xi_2)](\tau) && \text{by (5.10)} \\ &= \mathcal{F}_{012}[e^{it|\xi_3|^2} (\mathcal{F}_3 T_{12}\beta^{(k)})(t, x_1 - 2t\xi_3, x_2 - 2t\xi_3, \xi_3)](\tau, \xi_1, \xi_2) \end{aligned}$$

Let

$$(5.11) \qquad \sigma_{\xi_3}(t, x_1, x_2) \overset{\text{def}}{=} e^{it|\xi_3|^2} (\mathcal{F}_3 T_{12}\beta^{(k)})(t, x_1 - 2t\xi_3, x_2 - 2t\xi_3, \xi_3)$$

---

[8]It was first observed by X.C. [17] in the Hartree setting that the 6D retarded endpoint Strichartz estimate helps to deal with three-body interactions and shows that three-body interactions are "better" than two-body interactions. However, the problem we are discussing here provides a much deeper and much more substantial explanation to this phenomenon.

where ξ_3 is regarded as a fixed parameter. Then we have shown that

$$(5.12) \quad (\mathcal{F}_{012}\sigma_{\xi_3})(\tau, \xi_1, \xi_2) = (\mathcal{F}_{0123}\beta^{(k)})(\tau - |\xi_3|^2 + 2\xi_1 \cdot \xi_3 + 2\xi_2 \cdot \xi_3, \xi_1, \xi_2, \xi_3 - \xi_1 - \xi_2)$$

Now consider

$$\|\beta^{(k)}\|_{X_{-\frac{1}{2}+}} = \|\langle \tau + |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 \rangle^{-\frac{1}{2}+}(\mathcal{F}_{0123}\beta)(\tau, \xi_1, \xi_2, \xi_3)\|_{L^2_{\tau\xi_1\xi_2\xi_3}}$$

Change variable $\xi_3 \mapsto \xi_3 - \xi_1 - \xi_2$ and then $\tau \mapsto \tau - |\xi_3|^2 + 2\xi_1 \cdot \xi_3 + 2\xi_1 \cdot \xi_3$ and substitute (5.12) to obtain

$$\|\beta^{(k)}\|_{X_{-\frac{1}{2}+}} = \|\langle \tau + 2|\xi_1|^2 + 2|\xi_2|^2 + 2\xi_1 \cdot \xi_2 \rangle^{-\frac{1}{2}+}(\mathcal{F}_{012}\sigma_{\xi_3})(\tau, \xi_1, \xi_2)\|_{L^2_{\xi_3}L^2_{\tau\xi_1\xi_2}}$$

By the dual 6D endpoint Strichartz estimate [43]

$$\|\beta^{(k)}\|_{X_{-\frac{1}{2}+}} \lesssim \|\sigma_{\xi_3}\|_{L^2_{\xi_3}L^2_t L^{\frac{3}{2}+}_{x_1 x_2}}$$

Returning to the definition of σ_{ξ_3} given above in (5.11), and changing variable $x_1 \mapsto x_1 + 2t\xi_3$, $x_2 \mapsto x_2 + 2t\xi_3$, we obtain

$$\|\beta^{(k)}\|_{X_{-\frac{1}{2}+}} \lesssim \|\mathcal{F}_3 T_{12}\beta^{(k)}\|_{L^2_{\xi_3}L^2_t L^{\frac{3}{2}+}_{x_1 x_2}}$$

$\square$

Lemma 5.9 (Hölder and Sobolev). *If $\beta^{(k)}$ has any one of the following three forms*

$$(5.13) \qquad \beta^{(k)}(t, x_1, x_2, x_3) = \gamma^{(k)}(t, x_1, x_2, x_3) \times \begin{cases} V(x_1 - x_2)W(x_1 - x_3) \\ V(x_1 - x_2)W(x_2 - x_3) \\ V(x_1 - x_3)W(x_2 - x_3) \end{cases}$$

then all three of the following estimates hold
(5.14)

$$\|(\mathcal{F}_3 T_{12}\beta^{(k)})(t, x_1, x_2, \xi_3)\|_{L^2_t L^2_{\xi_3} L^{\frac{3}{2}+}_{x_1 x_2} L^2_c} \lesssim \begin{cases} \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|\langle \nabla_{x_1}\rangle^{\frac{3}{2}}\langle \nabla_{x_2}\rangle^{\frac{3}{2}}\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \\ \|V\|_{L^{2+}}\|W\|_{L^{2+}}\|\nabla_{x_1}\nabla_{x_2}\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \\ \|V\|_{L^{2+}}\|W\|_{L^{6+}}\|\nabla_{x_1}\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \\ \|V\|_{L^{6+}}\|W\|_{L^{2+}}\|\nabla_{x_2}\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \\ \|V\|_{L^{6+}}\|W\|_{L^{6+}}\|\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \end{cases}$$

(5.15)

$$\|(\mathcal{F}_2 T_{13}\beta^{(k)})(t, x_1, \xi_2, x_3)\|_{L^2_t L^2_{\xi_2} L^{\frac{3}{2}+}_{x_1 x_3} L^2_c} \lesssim \begin{cases} \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|\langle \nabla_{x_1}\rangle^{\frac{3}{2}}\langle \nabla_{x_3}\rangle^{\frac{3}{2}}\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \\ \|V\|_{L^{2+}}\|W\|_{L^{2+}}\|\nabla_{x_1}\nabla_{x_3}\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \\ \|V\|_{L^{2+}}\|W\|_{L^{6+}}\|\nabla_{x_1}\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \\ \|V\|_{L^{6+}}\|W\|_{L^{2+}}\|\nabla_{x_3}\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \\ \|V\|_{L^{6+}}\|W\|_{L^{6+}}\|\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{xx}'}} \end{cases}$$

(5.16)

$$\|(\mathcal{F}_1 T_{23}\beta^{(k)})(t,\xi_1,x_2,x_3)\|_{L_t^2 L_{\xi_1}^2 L_{x_2 x_3}^{\frac{3}{2}+} L_c^2} \lesssim \begin{cases} \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|\langle\nabla_{x_2}\rangle^{\frac{3}{2}}\langle\nabla_{x_3}\rangle^{\frac{3}{2}}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2} \\ \|V\|_{L^{2+}}\|W\|_{L^{2+}}\|\nabla_{x_2}\nabla_{x_3}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2} \\ \|V\|_{L^{6+}}\|W\|_{L^{2+}}\|\nabla_{x_2}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2} \\ \|V\|_{L^{2+}}\|W\|_{L^{6+}}\|\nabla_{x_3}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2} \\ \|V\|_{L^{6+}}\|W\|_{L^{6+}}\|\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2} \end{cases}$$

Proof. All of the estimates have a similar proof. As an illustrative example, consider the first estimate of (5.14). By (5.13),

$$(T_{12}\beta^{(k)})(t,x_1,x_2,x_3) = (T_{12}\gamma^{(k)})(t,x_1,x_2,x_3) \times \begin{cases} V(x_1-x_2)W(x_1) \\ V(x_1-x_2)W(x_2) \\ V(x_1)W(x_2) \end{cases}$$

Hence

$$(\mathcal{F}_3 T_{12}\beta^{(k)})(t,x_1,x_2,\xi_3) = (\mathcal{F}_3 T_{12}\gamma^{(k)})(t,x_1,x_2,\xi_3) \times \begin{cases} V(x_1-x_2)W(x_1) \\ V(x_1-x_2)W(x_2) \\ V(x_1)W(x_2) \end{cases}$$

By Hölder,

$$\|(\mathcal{F}_3 T_{12}\beta^{(k)})(t,x_1,x_2,\xi_3)\|_{L_{x_1 x_2}^{\frac{3}{2}+} L_c^2} \leq \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|(\mathcal{F}_3 T_{12}\gamma^{(k)})(t,x_1,x_2,\xi_3)\|_{L_{x_1 x_2}^{\infty-} L_c^2}$$

By Sobolev,

$$\|(\mathcal{F}_3 T_{12}\beta^{(k)})(t,x_1,x_2,\xi_3)\|_{L_{x_1 x_2}^{\frac{3}{2}+} L_c^2}$$
$$\leq \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{2}}\langle\nabla_{x_2}\rangle^{\frac{3}{2}}(\mathcal{F}_3 T_{12}\gamma^{(k)})(t,x_1,x_2,\xi_3)\|_{L_{x_1 x_2}^2 L_c^2}$$
$$= \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|\mathcal{F}_3\langle\nabla_{x_1}\rangle^{\frac{3}{2}}\langle\nabla_{x_2}\rangle^{\frac{3}{2}}(T_{12}\gamma^{(k)})(t,x_1,x_2,\xi_3)\|_{L_{x_1 x_2}^2 L_c^2}$$

Apply $L_{\xi_3}^2$ and use Plancherel,

$$\|(\mathcal{F}_3 T_{12}\beta^{(k)})(t,x_1,x_2,\xi_3)\|_{L_{\xi_3}^2 L_{x_1 x_2}^{\frac{3}{2}+} L_c^2}$$
$$= \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{2}}\langle\nabla_{x_2}\rangle^{\frac{3}{2}}(T_{12}\gamma^{(k)})(t,x_1,x_2,x_3)\|_{L_{\mathbf{xx'}}^2}$$
$$= \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{2}}\langle\nabla_{x_2}\rangle^{\frac{3}{2}}\gamma^{(k)}(t,x_1,x_2,x_3)\|_{L_{\mathbf{xx'}}^2}$$

The other estimates in (5.14) follow by using Hölder differently.   □

By splitting up $\gamma^{(k)}$ according to the relative magnitude of frequencies, we can share derivatives among three coordinates, as in the following corollary.

Corollary 5.10. *If $\gamma^{(k)}$ is symmetric and $\beta^{(k)}$ has any one of the following three forms*

$$(5.17) \qquad \beta^{(k)}(t, x_1, x_2, x_3) = \gamma^{(k)}(t, x_1, x_2, x_3) \times \begin{cases} V(x_1 - x_2)W(x_1 - x_3) \\ V(x_1 - x_2)W(x_2 - x_3) \\ V(x_1 - x_3)W(x_2 - x_3) \end{cases}$$

then

$$(5.18) \quad \|\beta^{(k)}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim \begin{cases} \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|\langle\nabla_{x_1}\rangle\langle\nabla_{x_2}\rangle\langle\nabla_{x_3}\rangle\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2} \\ \|V\|_{L^{2+}}\|W\|_{L^{2+}}\|\langle\nabla_{x_i}\rangle\langle\nabla_{x_j}\rangle\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2} \text{ with } i,j = 1,2,3 \text{ but } i \neq j \\ \|V\|_{L^{2+}}\|W\|_{L^{6+}}\|\langle\nabla_{x_i}\rangle\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2} \text{ with } i = 1,2,3 \\ \|V\|_{L^{6+}}\|W\|_{L^{6+}}\|\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2} \end{cases}$$

Proof. As in the proof of Corollary 5.6, it suffices to prove the first inequality of (5.18) since the other two follows directly from Lemma 5.9 and the fact that $\gamma^{(k)}$ is symmetric.

Split $\gamma^{(k)}$ according to whether $\max(|\xi_1|, |\xi_2|, |\xi_3|) = |\xi_3|$, $\max(|\xi_1|, |\xi_2|, |\xi_3|) = |\xi_2|$, or $\max(|\xi_1|, |\xi_2|, |\xi_3|) = |\xi_1|$

$$\gamma^{(k)} = P_{\substack{|\xi_1| \leq |\xi_3| \\ |\xi_2| \leq |\xi_3|}}\gamma^{(k)} + P_{\substack{|\xi_1| \leq |\xi_2| \\ |\xi_3| \leq |\xi_2|}}\gamma^{(k)} + P_{\substack{|\xi_2| \leq |\xi_1| \\ |\xi_3| \leq |\xi_1|}}\gamma^{(k)}$$

and define

$$\beta^{(k)}_{1,2\leq 3} \overset{\text{def}}{=} P_{\substack{|\xi_1| \leq |\xi_3| \\ |\xi_2| \leq |\xi_3|}}\gamma^{(k)} \times \begin{cases} V(x_1 - x_2)W(x_1 - x_3) \\ V(x_1 - x_2)W(x_2 - x_3) \\ V(x_1 - x_3)W(x_2 - x_3) \end{cases}$$

$$\beta^{(k)}_{1,3\leq 2} \overset{\text{def}}{=} P_{\substack{|\xi_1| \leq |\xi_2| \\ |\xi_3| \leq |\xi_2|}}\gamma^{(k)} \times \begin{cases} V(x_1 - x_2)W(x_1 - x_3) \\ V(x_1 - x_2)W(x_2 - x_3) \\ V(x_1 - x_3)W(x_2 - x_3) \end{cases}$$

$$\beta^{(k)}_{2,3\leq 1} \overset{\text{def}}{=} P_{\substack{|\xi_2| \leq |\xi_1| \\ |\xi_3| \leq |\xi_1|}}\gamma^{(k)} \times \begin{cases} V(x_1 - x_2)W(x_1 - x_3) \\ V(x_1 - x_2)W(x_2 - x_3) \\ V(x_1 - x_3)W(x_2 - x_3) \end{cases}$$

so that

$$\beta^{(k)} = \beta^{(k)}_{1,2\leq 3} + \beta^{(k)}_{1,3\leq 2} + \beta^{(k)}_{2,3\leq 1}$$

For the $\beta^{(k)}_{1,2\leq 3}$ piece, we use the first estimate of (5.8) combined with the first estimate of (5.14)

$$\|\beta^{(k)}_{1,2\leq 3}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|\langle\nabla_{x_1}\rangle^{\frac{3}{2}}\langle\nabla_{x_2}\rangle^{\frac{3}{2}}P_{\substack{|\xi_1| \leq |\xi_3| \\ |\xi_2| \leq |\xi_3|}}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2}$$

By the frequency restriction, we can move $\frac{1}{2}$ derivative in x_1 to x_3 and $\frac{1}{2}$ derivative in x_2 to x_3 to obtain:

$$\|\beta^{(k)}_{1,2\leq 3}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim \|V\|_{L^{\frac{3}{2}+}}\|W\|_{L^{\frac{3}{2}+}}\|\langle\nabla_{x_1}\rangle\langle\nabla_{x_2}\rangle\langle\nabla_{x_3}\rangle\gamma^{(k)}\|_{L_t^2 L_{\mathbf{xx'}}^2}$$

The term $\beta_{1,3\leq2}^{(k)}$ is handled analogously, using the second estimate of (5.8) together with the first estimate of (5.15). The term $\beta_{2,3\leq1}^{(k)}$ is handled using the third estimate of (5.8) together with the first estimate of (5.16). $\qquad\square$

Proposition 5.11. *For any $U(x)$, let $U_N(x) = N^{3\beta}U(N^\beta x)$. Then*

$$(5.19) \qquad \|R_{\leq M_3}^{(3)}(U_N(x_1 - x_2)U_N(x_1 - x_3)\,\alpha^{(3)})\|_{X_{-\frac{1}{2}+}^{(3)}} \lesssim C_U\|\alpha^{(3)}\|_{L_t^2 L_{\mathbf{x}_3\mathbf{x}_3}^2} \begin{cases} N^{5\beta+} \\ M_3^3 N^{2\beta+} \end{cases}$$

where

$$C_U = \|\nabla U\|_{L^{\frac{3}{2}+}}\|\nabla^2 U\|_{L^{\frac{3}{2}+}} + \|\nabla U\|_{L^{\frac{3}{2}+}}^2 + \|U\|_{L^{\frac{3}{2}+}}^2$$

Proof. To prove the top estimate of (5.19), we use do not use the frequency restriction and distribute all derivatives $\nabla_{x_1}\nabla_{x_2}\nabla_{x_3}\nabla_{x_1'}\nabla_{x_2'}\nabla_{x_3'}$ into the expression. The $\nabla_{x_1}'\nabla_{x_2'}\nabla_{x_3'}$ move directly onto $\alpha^{(3)}$. The expansion of

$$(5.20) \qquad \nabla_{x_1}\nabla_{x_2}\nabla_{x_3}\Big(U_N(x_1 - x_2)U_N(x_1 - x_3)(\nabla_{x_1'}\nabla_{x_2'}\nabla_{x_3'}\alpha_N^{(3)})\Big)$$

has $3 \times 2 \times 2 = 12$ terms total. Each is estimated using different estimates in (5.18). We will not write out each term, but take some representative examples. Let us consider the case

$$[\nabla_{x_1}\nabla_{x_2}\,U_N(x_1 - x_2)][\nabla_{x_3}\,U_N(x_1 - x_3)](\nabla_{x_1'}\nabla_{x_2'}\nabla_{x_3'}\alpha_N^{(3)})$$
$$= -N^{3\beta}(\nabla^2 U)_N(x_1 - x_2)(\nabla U)_N(x_1 - x_3)\alpha_N^{(3)}$$

Apply the first estimate of (5.18) to obtain

$$\|[\nabla_{x_1}\nabla_{x_2}\,U_N(x_1 - x_2)][\nabla_{x_3}\,U_N(x_1 - x_3)](\nabla_{x_1'}\nabla_{x_2'}\nabla_{x_3'}\alpha_N^{(3)})\|_{X_{-\frac{1}{2}+}}$$
$$\lesssim N^{3\beta}\|(\nabla^2 U)_N\|_{L^{\frac{3}{2}+}}\|(\nabla U)_N\|_{L^{\frac{3}{2}+}}\|S^{(3)}\alpha^{(3)}\|_{L_t^2 L_{\mathbf{x}_3\mathbf{x}_3'}^2}$$
$$\lesssim N^{5\beta+}\|\nabla^2 U\|_{L^{\frac{3}{2}+}}\|\nabla U\|_{L^{\frac{3}{2}+}}\|S^{(3)}\alpha^{(3)}\|_{L_t^2 L_{\mathbf{x}_3\mathbf{x}_3'}^2}$$

Another term resulting from the expansion of (5.20) is

$$U_N(x_1 - x_2)\,U_N(x_1 - x_3)(\nabla_{x_1}\nabla_{x_2}\nabla_{x_3}\nabla_{x_1'}\nabla_{x_2'}\nabla_{x_3'}\alpha_N^{(3)}))$$

In this case, we apply the fourth estimate of (5.18) to obtain

$$\|U_N(x_1 - x_2)\,U_N(x_1 - x_3)(\nabla_{x_1}\nabla_{x_2}\nabla_{x_3}\nabla_{x_1'}\nabla_{x_2'}\nabla_{x_3'}\alpha_N^{(3)}))\|_{X_{-\frac{1}{2}+}}$$
$$\lesssim \|U_N\|_{L^{6+}}^2\|S^{(3)}\alpha^{(3)}\|_{L_t^2 L_{\mathbf{x}_3\mathbf{x}_3'}^2}$$
$$\lesssim N^{5\beta+}\|U\|_{L^{6+}}^2\|S^{(3)}\alpha^{(3)}\|_{L_t^2 L_{\mathbf{x}_3\mathbf{x}_3'}^2}$$

To prove the bottom estimate of (5.19), we use the frequency restriction $R_{\leq M_3}^{(3)} \leq M_3^3 \nabla_{x_1'}\nabla_{x_2'}\nabla_{x_3'}$, and all of the $\nabla_{x_1'}\nabla_{x_2'}\nabla_{x_3'}$ derivatives move directly onto $\alpha^{(3)}$. One then estimates using the first estimate of (5.18) to obtain

$$\|R_{\leq M_3}^{(3)}(U_N(x_1 - x_2)U_N(x_1 - x_3)\,\alpha^{(3)})\|_{X_{-\frac{1}{2}+}^{(3)}} \lesssim M_3^3\|U_N\|_{L^{\frac{3}{2}+}}^2\|S^{(3)}\alpha_N^{(3)}\|_{L_t^2 L_{\mathbf{x}_3\mathbf{x}_3'}^2}$$
$$\lesssim M_3^3 N^{2\beta}\|U\|_{L^{\frac{3}{2}+}}^2\|S^{(3)}\alpha_N^{(3)}\|_{L_t^2 L_{\mathbf{x}_3\mathbf{x}_3'}^2}$$

$\square$

For the KIP estimates, we provide the following lemma and Proposition 5.13.

Lemma 5.12.

$$\left\|\int_{x_4} V(x_2 - x_4)W(x_3 - x_4)f(x_1 - x_4)\alpha^{(4)}(x_1, x_2, x_3, x_4; x_1', x_2', x_3', x_4)\, dx_4\right\|_{X^{(3)}_{-\frac{1}{2}+}}$$

$$(5.21)\quad \lesssim \begin{cases} \|V\|_{L^{1+}}\|W\|_{L^{\frac{3}{2}+}}\|f\|_{L^\infty}\|\langle\nabla_{x_3}\rangle\langle\nabla_{x_4}\rangle\langle\nabla_{x_4'}\rangle(\langle\nabla_{x_4}\rangle + \langle\nabla_{x_4'}\rangle)\alpha^{(4)}\|_{L^2_t L^2_{\mathbf{xx'}}} \\[2mm] \|V\|_{L^{1+}}\|W\|_{L^{\frac{3}{2}+}}\|f\|_{L^3}\|\langle\nabla_{x_1}\rangle\langle\nabla_{x_3}\rangle\langle\nabla_{x_4}\rangle\langle\nabla_{x_4'}\rangle(\langle\nabla_{x_4}\rangle + \langle\nabla_{x_4'}\rangle)\alpha^{(4)}\|_{L^2_t L^2_{\mathbf{xx'}}} \\[2mm] \|V\|_{L^{1+}}\|W\|_{L^{3+}}\|f\|_{L^\infty}\|\langle\nabla_{x_4}\rangle\langle\nabla_{x_4'}\rangle(\langle\nabla_{x_4}\rangle + \langle\nabla_{x_4'}\rangle)\alpha^{(4)}\|_{L^2_t L^2_{\mathbf{xx'}}} \\[2mm] \|V\|_{L^{1+}}\|W\|_{L^{3+}}\|f\|_{L^3}\|\langle\nabla_{x_1}\rangle\langle\nabla_{x_4}\rangle\langle\nabla_{x_4'}\rangle(\langle\nabla_{x_4}\rangle + \langle\nabla_{x_4'}\rangle)\alpha^{(4)}\|_{L^2_t L^2_{\mathbf{xx'}}} \end{cases}$$

Proof. Change variables $x_4 \mapsto x_4 + x_2$ to get

$$\beta^{(3)}(t, x_1, x_2, x_3; x_1', x_2', x_3')$$

$$\stackrel{\text{def}}{=} \int_{x_4} V(x_2 - x_4)W(x_3 - x_4)f(x_4 - x_1)\alpha^{(4)}(x_1, x_2, x_3, x_4; x_1', x_2', x_3', x_4)\, dx_4$$

$$= \int_{x_4} V(-x_4)W(x_3 - x_2 - x_4)f(x_4 + x_2 - x_1)\alpha^{(4)}(x_1, x_2, x_3, x_4 + x_2; x_1', x_2', x_3', x_4 + x_2)\, dx_4$$

Let us, for notational convenience, write

$$(5.22)\qquad \tilde{\sigma} \stackrel{\text{def}}{=} \alpha^{(4)}(x_1, x_2, x_3 + x_2, x_4 + x_2; x_1', x_2', x_3', x_4 + x_2)$$

$$(5.23)\qquad \begin{aligned} \sigma &\stackrel{\text{def}}{=} f(x_4 + x_2 - x_1)\tilde{\sigma} \\ &= f(x_4 + x_2 - x_1)\alpha^{(4)}(x_1, x_2, x_3 + x_2, x_4 + x_2; x_1', x_2', x_3', x_4 + x_2) \end{aligned}$$

Recalling that $T_3 g(x_2, x_3) = g(x_2, x_3 + x_2)$ (as in the proof of Lemma 5.4), we have

$$(T_3\beta^{(3)})(t, x_1, x_2, x_3; x_1', x_2', x_3') = \int_{x_4} V(-x_4)W(x_3 - x_4)\sigma\, dx_4$$

By (5.2),

$$\|\beta^{(3)}\|_{X^{(3)}_{-\frac{1}{2}+}} \lesssim \|\mathcal{F}_2 T_3\beta^{(2)}\|_{L^2_{\xi_2} L^2_t L^{\frac{6}{5}+}_{x_3} L^2_{x_1 x_1' x_2' x_3'}}$$

Moving the $L^2_{\xi_2}$ norm to the inside (by Minkowski's integral inequality) and applying Plancherel to convert $L^2_{\xi_2}$ to $L^2_{x_2}$

$$\|\beta^{(2)}\|_{X^{(3)}_{-\frac{1}{2}+}} \lesssim \|T_3\beta^{(2)}\|_{L^2_t L^{\frac{6}{5}+}_{x_3} L^2_{x_1 x_2 x_1' x_2' x_3'}}$$

By Minkowski's integral inequality,

$$\lesssim \int_{x_4} |V(-x_4)|\|W(x_3 - x_4)\,\sigma\|_{L^2_t L^{\frac{6}{5}+}_{x_3} L^2_{x_1 x_2 x_1' x_2' x_3'}}\, dx_4$$

$$= \int_{x_4} |V(-x_4)|\|W(x_3 - x_4)\,\|\sigma\|_{L^2_{x_1 x_2 x_1' x_2' x_3'}}\|_{L^2_t L^{\frac{6}{5}+}_{x_3}}\, dx_4$$

At this point, recalling (5.22), (5.23), we either estimate the inside term as

$$\|\sigma\|_{L^2_{x_1}} \leq \|f\|_{L^\infty} \|\tilde{\sigma}\|_{L^2_{x_1}} \tag{5.24}$$

which leads to the first and third estimates of (5.21) or we estimate using Hölder and Sobolev

$$\|\sigma\|_{L^2_{x_1}} \leq \|f\|_{L^3} \|\tilde{\sigma}\|_{L^6_{x_1}} \lesssim \|f\|_{L^3} \|\nabla_{x_1}\tilde{\sigma}\|_{L^2_{x_1}}$$

which leads to the second and fourth estimates of (5.21). Since the remaining steps are similar in either case, we will content ourselves to use (5.24) and prove the first and third estimates of (5.21) below.

We next apply Hölder in x_3. For the first estimate of (5.21), we use $\frac{5}{6} = \frac{2}{3} + \frac{1}{6}$, and for the third estimate of (5.21) we use $\frac{5}{6} = \frac{1}{3} + \frac{1}{2}$. Let us proceed with the proof of the first estimate in (5.21)

$$\lesssim \|W\|_{L^{\frac{3}{2}+}} \|f\|_{L^\infty} \int_{x_4} |V(-x_4)| \|\tilde{\sigma}\|_{L^2_t L^6_{x_3} L^2_{x_1 x_2 x_1' x_2' x_3'}} \, dx_4$$

$$\lesssim \|W\|_{L^{\frac{3}{2}+}} \|f\|_{L^\infty} \int_{x_4} |V(-x_4)| \|\tilde{\sigma}\|_{L^2_t L^2_{x_1 x_2 x_1' x_2' x_3'} L^6_{x_3}} \, dx_4$$

By Sobolev in x_3,

$$\lesssim \|W\|_{L^{\frac{3}{2}+}} \|f\|_{L^\infty} \int_{x_4} |W(-x_4)| \|\langle\nabla_{x_3}\rangle \tilde{\sigma}\|_{L^2_t L^2_{x_1 x_2 x_1' x_2' x_3'} L^2_{x_3}} \, dx_4$$

Now apply Hölder in x_4 to obtain

$$\lesssim \|W\|_{L^{\frac{3}{2}+}} \|V\|_{L^{1+}} \|f\|_{L^\infty} \|\langle\nabla_{x_3}\rangle\tilde{\sigma}\|_{L^{\infty-}_{x_4} L^2_{t x_2 x_3 x_2' x_3'}}$$

$$\lesssim \|W\|_{L^{\frac{3}{2}+}} \|V\|_{L^{1+}} \|f\|_{L^\infty} \|\langle\nabla_{x_3}\rangle\tilde{\sigma}\|_{L^2_{t x_2 x_3 x_2' x_3'} L^{\infty-}_{x_4}}$$

Apply Sobolev in x_4 to obtain

$$\lesssim \|W\|_{L^{\frac{3}{2}+}} \|V\|_{L^{1+}} \|f\|_{L^\infty} \|\langle\nabla_{x_4}\rangle^{\frac{3}{2}-}\langle\nabla_{x_3}\rangle \tilde{\sigma}\|_{L^2_{t x_2 x_3 x_2' x_3'} L^2_{x_4}}$$

Changing variable $x_3 \mapsto x_3 - x_2$ and $x_4 \mapsto x_4 - x_2$,

$$= \|W\|_{L^{\frac{3}{2}+}} \|V\|_{L^{1+}} \|f\|_{L^\infty} \|\langle\nabla_{x_4}\rangle^{\frac{3}{2}-}\langle\nabla_{x_3}\rangle\alpha^{(4)}(x_1, x_2, x_3, x_4; x_1', x_2', x_3', x_4)\|_{L^2_{t x_2 x_3 x_2' x_3'} L^2_{x_4}}$$

By standard trace estimates, we complete the proof of the first estimate in (5.21). $\quad\square$

Proposition 5.13.

$$\left\| R^{(3)}_{\leq M_3} \int_{x_4} V_N(x_2 - x_4) w_N(x_3 - x_4) f_N(x_1 - x_4)\alpha^{(4)} \, dx_4 \right\|_{X^{(3)}_{-\frac{1}{2}+}}$$

$$\lesssim C_{V, w_0, f}\left(N^{-\frac{1}{2}} \|S_4 S^{(4)}\alpha^{(4)}\|_{L^2_t L^2_{\mathbf{x_4 x_4'}}}\right) \begin{cases} M_3 N^{-\frac{1}{2}+} \\ N^{\beta - \frac{1}{2}+} \end{cases} \tag{5.25}$$

where $V_N(x) = N^{3\beta}V(N^\beta x)$, $w_N(x) = N^{\beta-1}w_0(N^\beta x)$, $f_N(x_1 - x_4) = w_0(N^\beta(x_1 - x_4))$ or $f_N(x_1 - x_4) = 1$, and

$$C_{V, w_0} = (\|V\|_{L^{1+}} + \|\nabla V\|_{L^{1+}} + \|w_0\|_{L^{3+}} + \|\nabla w_0\|_{L^{\frac{3}{2}+}})(1 + \|w_0\|_{L^\infty} + \|\nabla w_0\|_{L^3})$$

(which is finite and independent of N).

Proof. Let[9]

(5.26)

$$\beta^{(3)}(x_1, x_2, x_3; x_1', x_2', x_3') \overset{\text{def}}{=} R^{(3)}_{\leq M_3} \int_{x_4} V_N(x_2 - x_4) w_N(x_3 - x_4) f_N(x_1 - x_4) \alpha^{(4)} \, dx_4$$

$$= P^{(3)}_{\leq M_3} \nabla_{x_2} \nabla_{x_3} \int_{x_4} V_N(x_2 - x_4) w_N(x_3 - x_4) \nabla_{x_1} [f_N(x_1 - x_4) \nabla_{x_1'} \nabla_{x_2'} \nabla_{x_3'} \alpha^{(4)}] \, dx_4$$

If ∇_{x_1} lands on $f_N(x_1 - x_4)$, then we ultimately use the second or fourth estimate in (5.21). If, on the other hand, ∇_{x_1} lands on $\alpha^{(4)}$, then we ultimately use either the first or third estimate in (5.21). Since the two cases are similar, we will just proceed assuming that ∇_{x_1} lands on $\alpha^{(4)}$. Then (5.25) is the two estimates:

(5.27)
$$\|\beta^{(3)}\|_{X_{-\frac{1}{2}+}} \lesssim C_{V,w_0} \left(N^{-\frac{1}{2}} \|S_4 S^{(4)} \alpha^{(4)}\|_{L_t^2 L^2_{\mathbf{x_4 x_4'}}} \right) \begin{cases} M_2 N^{-\frac{1}{2}+} \\ N^{\beta - \frac{1}{2}+} \end{cases}$$

We begin by proving the first estimate in (5.27). Distributing the ∇_{x_3} derivative into the integral, we obtain two terms:

$$\beta^{(3)} = \text{A} + \text{B}$$

where

$$\text{A} = N^\beta P^{(3)}_{\leq M_3} \nabla_{x_2} \int_{x_4} V_N(x_2 - x_4)(\nabla_{x_3} w)_N(x_3 - x_4) f_N(x_1 - x_4) \nabla_{x_1} \nabla_{x_1'} \nabla_{x_2'} \nabla_{x_3'} \alpha^{(4)} \, dx_4$$

$$\text{B} = P^{(3)}_{\leq M_3} \nabla_{x_2} \int_{x_4} V_N(x_2 - x_4) w_N(x_3 - x_4) f_N(x_1 - x_4) \nabla_{x_1} \nabla_{x_3} \nabla_{x_1'} \nabla_{x_2'} \nabla_{x_3'} \alpha^{(4)} \, dx_4$$

Now use that $P^{(3)}_{\leq M_3} \nabla_{x_2} \leq M_3$ to obtain

$$\|\text{A}\|_{X_{-\frac{1}{2}+}} \lesssim M_3 N^\beta \left\| \int_{x_4} V_N(x_2 - x_4)(\nabla_{x_3} w)_N(x_3 - x_4) f_N(x_1 - x_4) \nabla_{x_1} \nabla_{x_1'} \nabla_{x_2'} \nabla_{x_3'} \alpha^{(4)} \, dx_4 \right\|_{X^{(2)}_{-\frac{1}{2}+}}$$

By the first estimate in (5.21),

$$\|\text{A}\|_{X_{-\frac{1}{2}+}} \lesssim M_3 N^{-1+} \|V\|_{L^{1+}} \|\nabla w_0\|_{L^{\frac{3}{2}+}} \|w_0\|_{L^\infty} \left\| (S^4 + S^{4'}) \frac{S^{(4)}}{\langle \nabla_{x_2} \rangle} \alpha^{(4)} \right\|_{L_t^2 L^2_{\mathbf{x_3 x_3'}}}$$

Again, using that $P^{(2)}_{\leq M_3} \nabla_{x_2} \leq M_3$, we obtain

$$\|\text{B}\|_{X_{-\frac{1}{2}+}} \lesssim M_3 \left\| \int_{x_4} V_N(x_2 - x_4) w_N(x_3 - x_4) f_N(x_1 - x_4) \nabla_{x_1} \nabla_{x_3} \nabla_{x_1'} \nabla_{x_2'} \nabla_{x_3'} \alpha^{(4)} \, dx_4 \right\|_{X^{(2)}_{-\frac{1}{2}+}}$$

By the third estimate in (5.21),

$$\|\text{B}\|_{X_{-\frac{1}{2}+}} \lesssim M_3 N^{-1+} \|V\|_{L^{1+}} \|w_0\|_{L^{3+}} \|w_0\|_{L^\infty} \left\| (S^4 + S^{4'}) \frac{S^{(4)}}{\langle \nabla_{x_2} \rangle} \alpha^{(4)} \right\|_{L_t^2 L^2_{\mathbf{x_3 x_3'}}}$$

Combining the above estimates for terms A and B, we obtain the first estimate of (5.27).

---

[9] We write the operator $R^{(k)}$ using true derivatives ∇ rather than $|\nabla|$. Once the $X_{-\frac{1}{2}+}$ norm is applied, one can be converted to the other.

For the second estimate in (5.27), starting from (5.26), we distribute both ∇_{x_2} and ∇_{x_3} into the integral. The result is four terms

$$\beta^{(2)} = \mathrm{C} + \mathrm{D} + \mathrm{E} + \mathrm{F}$$

where

$$\mathrm{C} = N^{2\beta} P^{(2)}_{\leq M_3} \int_{x_4} (\nabla V)_N(x_2 - x_4)(\nabla w)_N(x_3 - x_4) f_N(x_1 - x_4) \nabla_{x_1} \nabla_{x'_1} \nabla_{x'_2} \nabla_{x'_3} \alpha^{(4)} \, dx_4$$

$$\mathrm{D} = N^{\beta} P^{(2)}_{\leq M_3} \int_{x_4} V_N(x_2 - x_4)(\nabla w)_N(x_3 - x_4) f_N(x_1 - x_4) \nabla_{x_1} \nabla_{x_2} \nabla_{x'_1} \nabla_{x'_2} \nabla_{x'_3} \alpha^{(4)} \, dx_4$$

$$\mathrm{E} = N^{\beta} P^{(2)}_{\leq M_3} \int_{x_4} (\nabla V)_N(x_2 - x_4) w_N(x_3 - x_4) f_N(x_1 - x_4) \nabla_{x_1} \nabla_{x_3} \nabla_{x'_1} \nabla_{x'_2} \nabla_{x'_3} \alpha^{(4)} \, dx_4$$

$$\mathrm{F} = P^{(2)}_{\leq M_3} \int_{x_4} V_N(x_2 - x_4) w_N(x_3 - x_4) f_N(x_1 - x_4) \nabla_{x_1} \nabla_{x_2} \nabla_{x_3} \nabla_{x'_1} \nabla_{x'_2} \nabla_{x'_3} \alpha^{(4)} \, dx_4$$

For C and D, we use the first estimate of (5.21), and for E and F, we use the second estimate of (5.21). This gives

$$\|\mathrm{C}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim N^{\beta - \frac{1}{2}+} \|w_0\|_{L^\infty} \|\nabla V\|_{L^{1+}} \|\nabla w_0\|_{L^{\frac{3}{2}+}} \|w_0\|_{L^\infty} \left(N^{-\frac{1}{2}} \|(S_4 + S_{4'}) \frac{S^{(4)}}{\langle \nabla_{x_2} \rangle} \alpha^{(4)}\|_{L^2_t L^2_{\mathbf{x}_4 \mathbf{x}'_4}} \right)$$

$$\|\mathrm{D}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim N^{-\frac{1}{2}+} \|V\|_{L^{1+}} \|\nabla w_0\|_{L^{\frac{3}{2}+}} \|w_0\|_{L^\infty} \left(N^{-\frac{1}{2}} \|(S_4 + S_{4'}) S^{(4)} \alpha^{(4)}\|_{L^2_t L^2_{\mathbf{x}_4 \mathbf{x}'_4}} \right)$$

$$\|\mathrm{E}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim N^{\beta - \frac{1}{2}+} \|\nabla V\|_{L^{1+}} \|w_0\|_{L^{3+}} \|w_0\|_{L^\infty} \left(N^{-\frac{1}{2}} \|(S_4 + S_{4'}) \frac{S^{(4)}}{\langle \nabla_{x_2} \rangle} \alpha^{(4)}\|_{L^2_t L^2_{\mathbf{x}_4 \mathbf{x}'_4}} \right)$$

$$\|\mathrm{F}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim N^{-\frac{1}{2}+} \|V\|_{L^{1+}} \|w_0\|_{L^{3+}} \|w_0\|_{L^\infty} \left(N^{-\frac{1}{2}} \|(S_4 + S_{4'}) S^{(4)} \alpha^{(4)}\|_{L^2_t L^2_{\mathbf{x}_4 \mathbf{x}'_4}} \right)$$

Pulling these together gives the second estimate in (5.27). $\qquad\square$

Appendix A. The Topology on the Density Matrices

In this appendix, we define a topology τ_{prod} on the density matrices as was previously done in [28, 29, 30, 31, 32, 33, 45, 11, 18, 19, 20, 21, 22, 23, 24].

Denote the space of Hilbert-Schmidt operators on $L^2\left(\mathbb{R}^{3k}\right)$ as $\mathcal{L}^2_k$. Then $(\mathcal{L}^2_k)' = \mathcal{L}^2_k$. By the fact that $\mathcal{L}^2_k$ is separable, we select a dense countable subset $\{J^{(k)}_i\}_{i \geq 1} \subset \mathcal{L}^2_k$ in the unit ball of $\mathcal{L}^2_k$ (so $\|J^{(k)}_i\|_{\mathrm{op}} \leq 1$ where $\|\cdot\|_{\mathrm{op}}$ is the operator norm). For $\gamma^{(k)}, \tilde{\gamma}^{(k)} \in \mathcal{L}^2_k$, we then define a metric d_k on $\mathcal{L}^2_k$ by

$$d_k\big(\gamma^{(k)}, \tilde{\gamma}^{(k)}\big) = \sum_{i=1}^{\infty} 2^{-i} \left| \left\langle J^{(k)}_i, \big(\gamma^{(k)} - \tilde{\gamma}^{(k)}\big) \right\rangle \right|.$$

A uniformly bounded sequence $\gamma^{(k)}_N \in \mathcal{L}^2_k$ converges to $\gamma^{(k)} \in \mathcal{L}^2_k$ with respect to the weak topology if and only if

$$\lim_N d_k\big(\gamma^{(k)}_N, \gamma^{(k)}\big) = 0.$$

For fixed $T > 0$, let $C\left([0, T], \mathcal{L}_k^2\right)$ be the space of functions of $t \in [0, T]$ with values in $\mathcal{L}_k^2$ which are continuous with respect to the metric d_k. On $C\left([0, T], \mathcal{L}_k^2\right)$, we define the metric

$$\hat{d}_k(\gamma^{(k)}(\cdot), \tilde{\gamma}^{(k)}(\cdot)) = \sup_{t \in [0, T]} d_k(\gamma^{(k)}(t), \tilde{\gamma}^{(k)}(t)).$$

We can then define a topology τ_{prod} on the space $\oplus_{k \geqslant 1} C\left([0, T], \mathcal{L}_k^2\right)$ by the product of topologies generated by the metrics $\hat{d}_k$ on $C\left([0, T], \mathcal{L}_k^2\right)$.

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