Background on ∞-category theory. My reserach lies within the field of homotopy theory, with an especial focus on problems with an ∞-categorical flavor. Homotopy theory is the branch of mathematics concerned with structures arising from and applications of invariants which do not distinguish between "weakly equivalent" objects of some species – classically, topological spaces, as homotopy theory emerged from the field of algebraic topology. It is almost always desirable that structure-preserving functions between objects of interest induce corresponding functions between invariants – that is to say, in the language of category theory, that the invariants be functorial – and it was to formalize this behavior that category theory was introduced.

Because the weak equivalences which appear in homotopy theory are not in general isomorphisms in the 1-categorical sense but rather a proper generalization thereof, it can be useful to keep track of data relating the composite of a pair of "weakly inverse" morphisms to the respective identity morphisms – witnessing the fact that the pair were "weakly inverse" – and this leads directly to the eponymous homotopies between morphisms, to be conceptualized as "paths" connecting two morphisms. In fact, there arise further homotopies between homotopies and so on, all of which it is fruitful to keep track of. Grothendieck's celebrated homotopy hypothesis postulates that the correct structure for recording this data is a space, whether in the guise of a topological space, a simplicial set, or some other model, and categories in which the collections of parallel morphism carry this spatial structure are called ∞-categories in reference to the existence of homotopies between homotopies and so on.

The first technology for working with categories equipped with this homotopical structure was Quillen's model categories, introduced in [23], and an extremely robust theory thereof was developed in books such as [15], [16], and others. Beginning in the 1970s with Boardman and Vogt's "weak Kan complexes", introduced in [6], and especially since the beginning of the 21st century with, for example, [25] and [5], other technologies have been developed and compared to work with ∞-categories. These definitions have a reputation for abstruseness, but they have the payoff of significantly streamlining many proofs, and – once the foundations have been dealt with – largely succeed in restoring to ∞-category theory the sleekness enjoyed by many 1-category-theoretic arguments, and a great deal of ∞-category theory has been fleshed out in works such as [19], [18], [20], [26], [10], and others.

Background on representation stability and functor calculus. My dissertation research introduces a new flavor of functor calculus which is an extension of representation stability to stable ∞-category theory (note that the term "stable" regrettably has two distinct meanings here).

Denote by FI the category of finite sets and injections and by $\mathbb{Q}\mathbf{Vect}$ the category of rational vector spaces. A functor $\mathsf{FI} \to \mathbb{Q}\mathbf{Vect}$, called an FI-module, determines a sequence of representations of the symmetric groups $\mathfrak{S}_n$ by restricting the functor to automorphisms. Representation stability is a phenomenon enjoyed by many FI-modules of interest – especially including the cohomology of many moduli spaces and configuration spaces – which ensures that the representations determined by the FI-module eventually follow a certain predictable pattern. The theory has its origins in [8], was articulated in the language of FI-modules in [7], and in [9] the authors show that over Noetherian rings, an FI-module is representation stable and objectwise finite-dimensional if and only if it is finitely generated.

On the other hand, functor calculus refers to a family of techniques within homotopy theory concerned with approximating functors between certain categories by other, more well-behaved "polynomial" or "n-excisive" functors. The criterion for a "well-behaved" functor is always that it send certain diagrams in the domain category to limit diagrams. By far the most prominent member of this family of techniques is Goodwillie calculus, originally develped by Tom Goodwillie in [11], [12], and [13]. In its modern incarnation, captured in [18] and [4], Goodwillie calculus is concerned with functors between "differentiable" ∞-categories. Today, Goodwillie calculus has developed into a vast and fruitful subfield of homotopy theory with an array of important results and applications too numerous to list here.

Other flavors of functor calculus include orthogonal calculus, introduced by Michael Weiss in [29] and dealing with functors from the category of Euclidean spaces to topological spaces; embedding calculus, developed by Tom Goodwillie and Michael Weiss and introduced in [28] and dealing with space-valued presheaves on categories of manifolds and embeddings; and cotriple calculus, introduced by Brenda Johnson and Randy McCarthy in [17] and dealing with functors from pointed categories to abelian categories. It is to orthogonal calculus that my FI-calculus is most similar.

FI-calculus. Fix $\mathcal{V}$ a stable, presentable ∞-category. A stable ∞-category is the ∞-categorical analog of an abelian category, and presentability is a (co)completeness condition together with a set-theoretic tameness condition. I define a standard n-cube to be a diagram in FI, determined by a pair of sets $S \subseteq S'$ such that $|S' \setminus S| = n$ and consisting of all intermediate sets $S \subseteq T \subseteq S'$ along with the inclusion morphisms. I call a functor $\mathsf{FI} \to \mathcal{V}$ an "FI-object" and denote the ∞-category of FI-objects $\mathsf{FI}\mathcal{V}$. I define an *n-excisive* FI-object to be one sending all standard $n+1$-cubes to limit diagrams (also called "cartesian cubes") and denote the ∞-category of n-excisive FI-objects $\mathrm{Exc}_n\mathcal{V}$. I say that an FI-object E is *excisive* if there exists some $n \in \mathbb{N}$ such that $E \in \mathrm{Exc}_n\mathcal{V}$[1]. I show, in analogy to representation stability, that an FI-object is n-excisive if and only if it is left Kan extended from $\mathsf{FI}_{\leq n}$, the full subcategory of FI spanned by sets of cardinality at most n.

As in other flavors of functor calculus, there is a "Taylor tower" of universal n-excisive approximations $\mathbf{P}_nE$ under a given FI-object E. I call an FI-object E *n-homogeneous* if E is n-excisive and $\mathbf{P}_{n-1}E \cong 0$ and prove an equivalence

$$\mathrm{Homg}_n\mathcal{V} \simeq \mathfrak{S}_n\mathcal{V}$$

between the ∞-categories of n-homogeneous FI-objects and of $\mathfrak{S}_n$-objects in $\mathcal{V}$ – a result with direct analogs in orthogonal calculus and in Goodwilllie calculus. The layers of the Taylor tower of a given $E \in \mathsf{FI}\mathcal{V}$, the FI-objects

$$\mathbf{D}_nE \overset{\text{def}}{=} \mathrm{fib}\,\mathbf{P}_nE \to \mathbf{P}_{n-1}E$$

are n-homogeneous and hence determine $\mathfrak{S}_n$-objects, which I call the *Taylor coefficients* $\mathbf{C}_nE$ of E.

Surprisingly, there exist natural transformations between the Taylor coefficients of an FI-object making those coefficients – a priori only a symmetric sequence – into an FI-object themselves. The first main question of my dissertation is to establish how much information can be recovered from these Taylor coefficients along with their FI-object structure. To address this question in full generality, I introduce the ∞-category

$$\mathrm{ExSeq}\mathcal{V} \overset{\text{def}}{=} \lim \cdots \xrightarrow{\mathbf{P}_n} \mathrm{Exc}_n\mathcal{V} \xrightarrow{\mathbf{P}_{n-1}} \cdots \xrightarrow{\mathbf{P}_0} \mathrm{Exc}_0\mathcal{V}$$

of "formal Taylor towers," which I call *excision sequences*. $\mathrm{ExSeq}\mathcal{V}$ is the natural domain of the aggregate Taylor coefficient functor $\mathbf{C}$, and I prove that $\mathbf{C}$ determines an equivalence of ∞-categories

$$\mathbf{C} : \mathrm{ExSeq}\mathcal{V} \simeq \mathsf{FI}\mathcal{V}$$

I also prove that any FI-object that is an iterated limit of excisive FI-objects is the limit of its own Taylor tower, and I call such FI-objects analytic and denote the ∞-category of these $\mathsf{FI}\mathcal{V}^{\mathrm{Anly}}$. I obtain a restricted equivalence

$$\mathbf{C} : \mathsf{FI}\mathcal{V}^{\mathrm{Anly}} \simeq \mathsf{FI}\mathcal{V}^{\mathrm{Tors}}$$

where $\mathsf{FI}\mathcal{V}^{\mathrm{Tors}}$ consists of those FI-objects which are colimits of finitely supported FI-objects.

My second main result deals with the specialization to the case $\mathcal{V} = \mathcal{S}p^{\mathbb{Q}}$ of functors from FI to the ∞-category of rational chain complexes and establishes FI-calculus as a direct generalization of representation stability to the setting of stable ∞-categories. I show that if an FI-chain complex E is n-excisive for some $n \in \mathbb{N}$, then its homology is representation stable, that if an FI-module $E : \mathsf{FI} \to \mathbb{Q}\mathbf{Vect}$ is representation stable, then there exists $n \in \mathbb{N}$ such that, when E is considered as a discrete FI-chain complex, E agrees with $\mathbf{P}_nE$ outside of a finite range, and finally that the $\mathfrak{S}_n$-representations appearing in the stable range of the homology of an n-excisive FI-chain complex E can be directly read off from the homology of the coefficient FI-chain complex $\mathbf{C}E$. These results largely follow from an explicit computation of $\mathbf{D}_nF_n$, where $F_n \overset{\text{def}}{=} \mathbb{Q}[\mathsf{FI}(n, -)]$, in terms of a combinatorially defined poset.

[1] This conflicts with some literature in which "excisive" is a synonym for "1-excisive".

Relation to other work. In [27], Steven Sam and Andrew Snowden prove an equivalence between the derived categories of tails of finitely generated rational FI-modules and of finite-dimensional, finitely supported rational FI-modules. In [22], Peter Patzt and John Wiltshire-Gordon develop new invariants which form a category FJ and show that for any commutative ring R, the category of tails of representation stable FI-R-modules is equivalent to the category of finitely supported FJ-R-modules. Patzt and Wiltshire-Gordon's work generalizes that of Sam and Snowden by dropping the finiteness condition, allowing for $\mathbb{Q}$ to be repalced by any commutative ring, and establishing an equivalence not of derived categories but of abelian categories. This comes at the cost of using the category FJ, which is much significantly more complicated than FI.

My result that

$$\mathbf{C} : \mathsf{FI}\mathcal{V}^{\mathrm{Anly}} \simeq \mathsf{FI}\mathcal{V}^{\mathrm{Tors}}$$

is a direct generalization of that of Sam and Snowden, who seem to construct the coefficient functor in a different guise. My result can also be viewed as a strengthening of that of Patzt and Wiltshire-Gordon, although it is not a direct generalization, because Patzt and Wiltshire-Gordon's tail invariants are different from my Taylor coefficients. Like Patzt and Wiltshire-Gordon, my result generalizes that of Sam and Snowden by removing the finiteness requirements, considering functors from FI to categories other than $\mathbb{Q}\mathbf{Vect}$, and lifting an equivalence of derived categories to an equivalence of ∞-categories (Patzt and Wiltshire-Gordon do not use ∞-categories, but their equivalence of abelian categories yields an equivalence of ∞-categories of chain complexes). Also like Patzt and Wiltshire-Gordon, I prove an explicit dictionary translating between $\mathfrak{S}_n$-representations appearing in the stable range of an FI-module and its FI-module of coefficients (FJ-module of invariants, respectively).

My result also extends that of Patzt and Wiltshire-Gordon by considering not just excisive FI-objects (which are essentially generalizations of representation stable FI-modules) but analytic ones and by allowing for FI-objects which take values not just in ∞-categories of chain compexes but in ∞-categories of spectra or in any other presentable stable ∞-category, thereby facilitating applications involving extraordinary cohomology theories. My result also benefits from using a much simpler category of invariants – FI – than that used by Patzt and Wiltshire-Gordon – FJ.

The extension from excisive FI-objects to analytic FI-objects in particular is significant because every FI-object whatsoever admits a universal approximation by an analytic FI-object, which can in turn be fully described by its FI-object of Taylor coefficients. This means that the technology of my FI-calculus can be applied to any FI-object – even one whose homology never satisfies representation stability – to describe its long-term representation-theoretic behavior.

Moreover, it is usually of interest to give explicit bounds on the stable range of a representation stable FI-module and sometimes also to calculate the behavior of a representation stable FI-module outside of its stable range. An FI-object that is n-excisive may have a stable range in the sense of representation stability that does not begin until $2n$. But such FI-objects should nonetheless be considered as "stable" as can be, and their behavior before their "stable range" can still be determined from their Taylor coefficients. This suggests that in addition to calculating the stable ranges of the homology of an FI-chain complex or FI-spectrum that it may be useful to consider the discrepancy between an FI-object and its analytic approximation. It may be the case that some FI-objects of interest are much more well-behaved in the sense of agreeing with their analytic approxmations (and therefore being well-described by their Taylor coefficients) than can be seen just by considering the stable ranges of their homology FI-modules. This approach represents a generalization of the "higher-order representation stability" introduced by Jeremy Miller and Jennifer Wilson in [21].

Future directions. Another of the chief benefits of the ∞-categorical perspective on representation stability is the wide range of new avenues it opens up for further research. To start, FI-calculus is merely the terminal example in a vast family of functor calculi which can be set up using my methods. Given any Cartesian fibration

$$\varpi : \mathcal{D} \to \mathsf{FI}$$

it is possible to define standard n-cubes in $\mathcal{D}$, giving rise to n-excisive functors, Taylor towers, and Taylor coefficients

$$\mathbf{C}_n E : \varpi^{-1}(\mathfrak{S}_n) \to \mathcal{V}$$

Examples include ∞-categories of totally ordered finite sets, partially ordered finite sets, cyclically ordered finite sets, and directed or undirected graphs, as well as more involved examples: e.g. given a manifold M, the ∞-category $M\mathbf{Braid}$ with objects finite sets of distinct marked points in M and morphisms given by braids from one set of points to another. When $M = \mathbb{R}^2$, this recovers the category of braids which is captured as an example by Oscar Randal-Williams and Natalie Wahl in [24]. Further, when ϖ has a symmetric monoidal structure compatible with its Cartesian structure, we can refine our notion of excision so that degrees of excisiveness are indexed not just by natural numbers but by sets of objects of $\mathcal{D}$. I am eager to investigate the structure carried by the generalized Taylor coefficients that arise in these settings and to determine to what extent these generalized Taylor coefficients succeed in describing analytic functors $\mathcal{D} \to \mathcal{V}$.

The ∞-categorical perspective on representation stability also invites investigation of the interactions between representation stability and Goodwillie calculus. In [3], David Barnes and Rosona Eldred investigate the interaction between orthogonal calculus and Goodwillie calculus by composing functors of interest in Goodwillie calculus with the functor

$$V \mapsto S^V : \mathcal{J} \to \mathcal{S}$$

sending a vector space to its one-point compactification, where $\mathcal{J}$ is the ∞-category of Euclidean spaces and $\mathcal{S}$ is the ∞-category of topological spaces. I am interested in similar questions: given stable presentable ∞-categories $\mathcal{V}$ and $\mathcal{W}$ and functors

$$E : \mathsf{FI} \to \mathcal{V}$$

and

$$F : \mathcal{V} \to \mathcal{W}$$

how do the Taylor coefficients of E and F (in the FI and Goodwillie sense respectively) relate to the coefficients of $F \circ E$? Is there some sort of chain rule that describes their relationship? Is there any relationship between such a chain rule and that described by Greg Arone and Michael Ching in [2]? Going in the other direction, given a functor

$$G : \mathsf{FI} \to \mathcal{W}$$

we can ask what the Taylor coefficients of E and G tell us about the Goodwillie tower of $\mathrm{Lan}_E\, G$.

Another approach to blending FI-calculus with Goodwillie calculus is to replace the condition that the codomain category $\mathcal{V}$ be stable with the requirement that it be n-excisive for some $n \in \mathbb{N}$ in the sense of Heuts in [14]. As n increases, this allows for us to approach the general, unstable case while retaining enough tameness in the codomain category that much of the technology developed in my dissertation will still apply. When $\mathcal{V}$ is an ∞-topos (also called an ∞-logos in this context), I strongly expect that FI-calculus fits into the framework of "generalized Goodwillie towers" developed by Mathieu Anel, Georg Biedermann, Eric Finster, and Andre Joyal in forthcoming work. This would immediately establish that the ∞-categories $\mathrm{Homg}_n \mathcal{V}$ are not-necessarily-pointed stable ∞-categories, and one would hope to establish a reconstruction theorem analogous to that of Arone and Ching in [1].

I plan to carry out the research outlined above in the immediate future. Ideally, each phase will integrate the last, so that I ultimately obtain a functor calculus for functors $\mathcal{D} \to \mathcal{V}$ for $\mathcal{V}$ not necessarily stable along with theorems relating this functor calculus to Goodwillie calculus. Furthermore, as previously discussed, FI-calculus is highly analogous to Weiss' orthogonal calculus – a somewhat fanciful interpretation is that FI-calculus is orthogonal calculus with the field $\mathbb{R}$ of the rationals replaced by $\mathbb{F}_1$, the speculative field with one element. In my dissertation, I prove results which do not have known analogs in orthogonal calculus, but which could plausibly be extended to that setting. I intend to explore these possibilities in the hopes of deepending the theory of orthogonal calculus with insights and inspirations from FI-calculus.

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