## MATH 285y TROPICAL GEOMETRY SPRING 2013
## PROBLEM SET 2, DUE TUESDAY MARCH 5

1. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

be a matrix over $K = \mathbb{C}\{\{t\}\}$ and let $M$ be the matroid of $A$, with ground set $\{0, 1, 2, 3, 4, 5\}$ corresponding to the columns of $A$.

(a) As in class, let $J$ be the homogeneous linear ideal in $K[x_0, \ldots, x_5]$ whose projective variety is the rowspan of $A$ in $\mathbb{P}^5$. Compute $J$. List the circuits of $J$ and of $M$, respectively.

(b) Draw the Hasse diagram of the lattice of flats of $M$, and show that the flats of $M$ are in correspondence with partitions of the set $\{1, 2, 3, 4\}$.

(c) Sidenote: given the lattice of flats of a matroid $M$, how would you recover the data of its circuits; its independent sets; its bases?

(d) As in class, let $I$ be the image of $J|_{x_0=1}$ in $K[x_1^{\pm}, \ldots, x_5^{\pm}]$. Draw the tropical variety of $I$ as well as you can; prove that it is homeomorphic to a cone over the Petersen graph.

(e) For each $i = 0, \ldots, 5$, let $H_i$ be the plane in $\mathbb{P}^3_K$ with normal vector $a_i$, where $a_i$ denotes the $i^{th}$ column of $A$. Let $X = \mathbb{P}^3_K \setminus \cup H_i$ be the complement. Show that the map $X \to (K^*)^5$ sending

$$x = (x_0 : x_1 : x_2 : x_3) \mapsto (a_0 \cdot x : \cdots : a_5 \cdot x) \in (K^*)^6/K^* = (K^*)^5$$

is injective, and identify its image with $V(I)$. Check that the collection of intersections of the hyperplanes $H_i$, ordered by inclusion, form a partially ordered set that is dual to the lattice of flats of $M(A)$. (Thus the tropical variety records the combinatorics of "what's missing" from $X$.)

2. We saw in class that $G_{2,4}$ modulo its lineality space is a 1-dimensional complex consisting of 1 vertex and 3 rays, corresponding to the four different ways that a tropical Plücker vector $(P_{12}, \ldots, P_{34}) \in \mathbb{R}^6$ can achieve the minimum among

$$\{P_{12} + P_{34}, P_{13} + P_{24}, P_{14} + P_{23}\}$$

at least twice.

For each of these four combinatorial types, give an example of a $2 \times 4$ matrix whose tropical Plücker vector achieves this type, and draw a picture of the corresponding tropicalized line in $\mathbb{R}^3$.

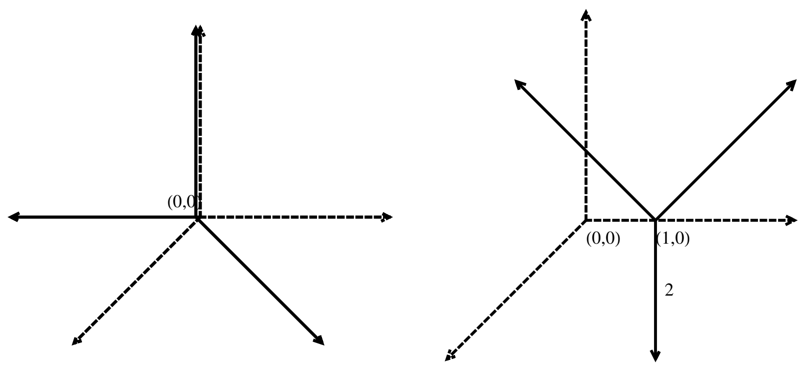
To draw these pictures, it could possibly help to note that the circuits of any linear ideal form a tropical basis. We proved this in the case of valuation 0 coefficients; see 4.2 of Speyer's "Tropical linear spaces" for the general case. Alternatively, now is a great time to familiarize yourself with Anders Jensen's software `gfan`, whose basic functionality can be accessed through `sage`.

3. Given finitely many polynomials $f_1, \ldots, f_s \in \mathbb{Q}[x_1, \ldots, x_n]$, how would you compute whether the ideal $I = \langle f_1, \ldots, f_s \rangle$ contains a monomial? Your description should be specific enough that it could be implemented in a computational algebra package like `Macaulay2`.

4. Extend tropical Bézout's theorem to stable intersections as follows.

   (a) Prove that the *stable intersection* $C \cap_{stab} D = lim_{\epsilon \to 0} C \cap (D + \epsilon v)$ of two tropical plane curves is well-defined, i.e. independent of the choice of a generic vector $v$.

   (b) Define the multiplicity of a point $p$ in the stable intersection of two tropical plane curves $C \cap_{stab} D$; ensure that your definition is independent of any choice of perturbation. Try your definition on each of the intersections below.




   (c) Deduce immediately, using last week's homework, that two plane tropical curves $C$ and $D$ of degrees $c$ and $d$ stably intersect in $c \cdot d$ points, counted with multiplicity.

5. Given a (trivially valued) field $K$, a polynomial $f = \sum c_u x^u \in K[x_1^{\pm}, \dots, x_n^{\pm}]$, and $w \in \mathbb{R}^n$, recall that $\text{in}_w(f) = \sum c_{u_i} x^{u_i}$, where the sum is over all $u_i \in \mathbb{Z}^n$ such that $w \cdot u_i$ is minimal. This exercise outlines how to compute $\text{in}_w(I)$.

   (a) Given any monomial ordering $\prec$, show that $\text{in}_\prec \text{in}_w I = \text{in}_{\prec_w} I$.

   (b) Show that if $\{g_1, \dots, g_s\}$ is a Gröbner basis for $I$ with respect to $\prec_w$, then $\text{in}_w g_i$ is a Gröbner basis for $\text{in}_w I$ with respect to $\prec$.

   (c) Conclude that $\text{in}_w I = \langle \text{in}_w g_1, \dots, \text{in}_w g_s \rangle$.