Problem Set 1. Due Friday September 15 in class

1. Eisenbud-Harris exercises I-1, I-2, I-3 (graded for completion only)

2. Classify, with proof, the points of \( \text{Spec } \mathbb{C}[x, y]/(xy) \), and describe the closure of each point in the Zariski topology.

3. Do the same for \( \text{Spec } (\mathbb{C}[x, y]/(xy))_{(x,y)} \).

4. Do the same for \( \text{Spec } \mathbb{Z}[x] \) (see Eisenbud-Harris II-37, II-38).

5. Liu exercise 2.1.6.

Problem Set 2. Due Friday September 22 in class

1. Liu exercises 2.1.2, 2.1.3, 2.1.4, 2.2.1, 2.2.2. See online errata for 2.1.4(a): replace “nilpotent” with “nil ideal”

Problem Set 3. Due Friday September 29 in class

1. Eisenbud-Harris exercise I-20

2. Liu exercises 2.2.4, 2.2.8, 2.2.9, and either 2.2.6 or 2.2.13.

Problem Set 4. Due Friday October 6 in class

1. We studied a once-punctured plane \( \mathbb{A}^2_k - \{(0,0)\} \) in class. Consider a twice-punctured plane. Is it isomorphic as a scheme to a once-punctured plane?

2. Verify that the disjoint union of finitely many affine schemes is an affine scheme.

3. Liu exercises 2.3.1, 2.3.14, 2.3.15.

Problem Set 5. Due Friday October 13 in class


2. Eisenbud-Harris exercise I-46, just parts (d) through (g)

3. Liu exercises 2.3.7, 3.1.6, 3.1.8.
Problem Set 6. Due Friday October 20 in class

2. Liu exercises 2.4.1, 2.4.2 (see example 2.3.16), and 2.4.3.

Problem Set 7. Due Friday October 27 in class

1. Let $B = \text{Spec } \mathbb{C}[t]$. Using a computer or otherwise, compute the limits of the following families of closed subschemes of $\mathbb{A}^n_B$ as $t \to 0$, in the precise sense discussed. What are the primary components of the limiting scheme? Draw pictures.
   
   (a) The plane curve $xy^2 = t$

   (b) Three concurrent lines becoming coplanar: the three lines through the origin and 
   
   $(1, 0, 0), (0, 1, 0), (1, 1, t)$ respectively

   (c) Squashing a twisted cubic curve: the space curve whose closed points are $(ts, s^2, s^3)$ for $t, s \in \mathbb{C}$.

2. Liu exercises 2.4.4, 2.4.9, 2.4.11, 2.5.3.

Problem Set 8. Due Friday November 3 in class

1. Liu exercises 2.3.10, 2.3.11, 2.3.18, 2.5.7, 3.1.5.

Problem Set 9. Due Friday November 10 in class

1. Vakil exercise 8.2.N (see 8.2.11)

2. Let $d, n$ be positive integers, and let $N = \binom{n+d}{d} - 1$. Prove that the image of the Veronese map $v_d(\mathbb{P}^n_\mathbb{C}) \subset \mathbb{P}^N_\mathbb{C}$ does not lie on any hyperplane (i.e. vanishing locus of a linear form) of $\mathbb{P}^N_\mathbb{C}$.

3. (You may replace $\text{Gr}(2, 4)$ with $\text{Gr}(d, n)$ as you wish.)
   
   Consider the Grassmannian variety $\text{Gr}(2, 4)$ parametrizing 2-planes in $\mathbb{C}^4$. For each 2-element subset $I$ of $\{1, \ldots, 4\}$, consider all $2 \times 4$ matrices with reduced row echelon form having leading 1s in columns $I$.

   (a) Show that the row spans of such matrices form the closed points of a locally closed—i.e., intersection of closed and open—subvariety $\Sigma_I$ of $\text{Gr}(2, 4)$, and that each $\Sigma_I$ is isomorphic to an affine space (of what dimension?)

   (b) Argue that $\bigcup_{I} \Sigma_I = \bigcup_{I'} \Sigma_{I'}$ for appropriately chosen $I'$. Partially order the $\Sigma_I$ according to whether the closure of one contains the other.

4. Liu exercise 3.2.6.
Problem Set 10. Due Friday November 17 in class

1. Practice the valuative criterion: use it to verify that $\mathbb{A}_k^n$ is proper over $k$ if and only if $n = 0$.

2. Over $\mathbb{C}$, argue that the polynomial

$$F(x) = x^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$$

has at most 3 distinct roots iff $F$ and $F'$ have a common factor of degree 3. Set up a Macaulay2 computation to find equations for the closed set of $\mathbb{A}^6$ consisting of those points $(b, c, d, e, f, g)$ such that $F(x)$ has at most 3 distinct roots\(^1\)

3. Liu exercise 3.3.12

4. Prove the statement in Liu exercise 3.3.15, using the suggested method or otherwise.

Problem Set 11. Due Friday December 1 in class

1. List, with justification, the points of $\text{Spec } \mathbb{R}[x, y]/(x^2 + y^2)$.

2. Eisenbud-Harris exercises II-6, II-7.

3. Liu exercises 3.2.9, 4.2.7.

Problem Set 12. Due Monday December 11 in class

1. Vakil exercise 5.4.H.

2. Liu exercises 4.2.10, 4.2.11, and 4.2.12. For 4.2.10, note “Euler’s Lemma,” that a homogeneous degree $d$ polynomial $F \in k[T_0, \ldots, T_n]$ satisfies $\sum T_i \frac{\partial F}{\partial T_i} = d \cdot F$.

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\(^1\) If you actually want to try it, here is some sample Macaulay2 code that may be helpful.

```macaulay2
R=QQ[a..f];
M=matrix{{a,b,c},{d,e,f}};
minors(2,M)
```