ALGEBRA PRESENTATION WRITEUP

CORY GRIFFITH

1. Vector Field Handwaving

We first make some thoroughly non-rigorous and nonstandard definitions.

Definition 1.1. Let X be a smooth manifold. A *vector field* is a smooth assignment of a vector to each point. A *vector bundle* is a module of vector fields, with the coefficient ring being the smooth functions.

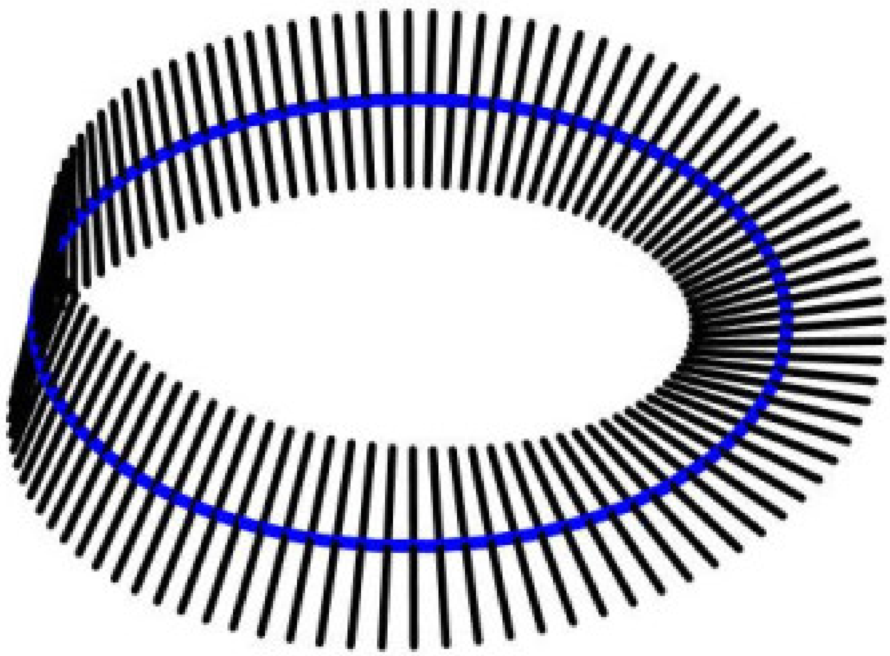
We think of a vector bundle as a rule for what vectors we're allowed to assign to points.

Example 1.2. The *tangent bundle* of X consists of all vector fields for which the vector at each point is tangent to X.

Example 1.3. Consider $\mathbb{R}^2$. We can take our rule to be the assignment of a vector in $\mathbb{R}^n$ to each point. When $n = 2$, we get the familiar vector fields on $\mathbb{R}^2$ of calculus. Something interesting happens when $n = 1$: the information of a vector field is exactly that of a smooth function, so our module is $C^\infty(\mathbb{R}^2)$ as a module over itself.

Definition 1.4. When the vector fields of a vector bundle have one dimension of vectors allowed at each point, we call the vector bundle a *line bundle*. When the definitions are rigorously set up, it is immediate that being a line bundle is equivalent to the following property: for any small enough open set, the information of a vector field on that open set formed according to the rules of our bundle is exactly the information of a smooth function.

Example 1.5. When we consider the circle S^1, there are two line bundles to keep in mind. The first is the tangent bundle. The second can be obtained as follows: stick S^1 into a Mobius band, running around the middle.



We let the vector fields consist of a choice of vector that points along the mobius band perpendicular to the circle. See the illustration (stolen off the internet) - the vectors are allowed to go in the directions of the black lines. We will call the resulting module $\mathbb{M}^T$.

Because of the twisting of the vector bundle, no vector field associated with this bundle avoids having a zero vector. On the other hand, if we consider a smaller open set of the circle, with the same rule, we can 'untwist' and see that vector fields on this smaller open set are equivalent to smooth functions.

Example 1.6. A more sophisticated pair of examples are given by considering projective n-space $\mathbb{P}^n_{\mathbb{R}}$. We obtain $\mathbb{P}^n_{\mathbb{R}}$ by collapsing each line through the origin in $\mathbb{R}^{n+1}$ to a point. There is thus a natural line bundle called $\mathcal{O}(-1)$, the *tautological bundle*, defined by the rule that to each point in $\mathbb{P}^n_{\mathbb{R}}$ we may assign a vector lying in the line that was collapsed to it. Alternatively, we can require that to each point we assign a linear functional on the line corresponding to that point - that is, to each point, we assign a dual vector (we will not address the question of what a 'smooth choice of dual vectors' actually means). This is referred to as $\mathcal{O}(1)$.

2. On Affine Schemes

Here, we follow Eisenbud.

Modules over a ring R serve the role of generalized vector bundles on Spec R. Some are more like vector bundles than others. Those that are analogous to line bundles are called (in the parlance of Eisenbud - this terminology is not widespread) invertible modules.

Definition 2.1. A finite R-module I is called *invertible* if for every prime P of R, we have an isomorphism of R_P modules $I_P \cong R_P$.

To see the analogy with line bundles, remember that R_P is the collection of functions near the point P in $\operatorname{Spec} R$.

Definition 2.2. For any R-module M, let $M^* := \operatorname{Hom}_R(M, R)$.

While $\mathbb{M}^T$ failed to be isomorphic to the module of smooth functions, it turns out there is a good way to 'untwist' it, or any line bundle. An analogue of the following theorem is usually proven purely geometrically, in the much more general context of locally ringed spaces, from which the algebraic result here follows as a corollary.

Theorem 2.3. *Let R be a Noetherian ring. An R-module I is invertible if and only if the map $\mu : I^* \otimes I \to R$ with $\varphi \otimes m \mapsto \varphi(m)$ is an isomorphism.*

Proof. Recall that a map of modules is injective (or surjective) if and only if it is locally injective (or surjective): for each prime P it is injective (or surjective) after localizing at P.

Suppose that I is invertible. Pick any prime P and let $\iota : R_P \to I_P$ be an isomorphism. Since localization distributes across tensor products, we have $(I^* \otimes I)_P = I_P^* \otimes I_P$. Then we have the following sequence of isomorphisms.

$$\begin{array}{ccccccc} I_P^* \otimes I_P & \xrightarrow{\sim} & R_P^* \otimes R_P & \xrightarrow{\sim} & R_P \otimes R_P & \xrightarrow{\sim} & R_P \\ \\ \varphi \otimes m & \longmapsto & (\varphi\iota) \otimes \iota^{-1}(m) & \longmapsto & (\varphi\iota(1)) \otimes \iota^{-1}(m) & \longmapsto & \iota^{-1}(m) \cdot \varphi(\iota(1)) \end{array}$$

Since $\iota^{-1}(m) \cdot \varphi(\iota(1)) = \varphi(\iota(\iota^{-1}(m)) = \varphi(m)$, the composition is exactly μ, so μ is an isomorphism.

Conversely, suppose that μ is an isomorphism. We wish to exhibit an isomorphism $I_P \cong R_P$ for each P, and then show that I is finitely generated. Let $1 = \mu(\sum_i \varphi_i \otimes a_i)$. Since $1 \notin P$, we must have that some $\varphi_i(a_i) \notin P$. Letting $a \in I_P$ be $(1/\varphi_i(a_i)) \cdot a_i$, we have $\varphi_i(a) = 1$, from which it is immediate that φ_i is surjective. Letting $\sigma : I_P \to R_P$ be evaluation at a, that $\sigma(\varphi_i) = 1$ shows σ is surjective, so we obtain two short exact sequences.

$$0 \to \ker \varphi_i \to I_P \to R_P \to 0$$

$$0 \to \ker \sigma \to I_P^* \to R_P \to 0$$

Splitting the first sequence by $1 \mapsto a$ and the second by $1 \mapsto \varphi_i$ demonstrates that we have decompositions $I_P = \ker \varphi_i \oplus R_P \cdot a$ and $I_P^* = R_P \cdot \varphi_i \oplus \ker \sigma$. Thus

$$I_P^* \otimes I_P \cong (R_P\varphi_i) \otimes \ker \varphi_i \oplus (R_P\varphi_i) \otimes (R_P a) \oplus \ker \sigma \otimes \ker \varphi_i \oplus \ker \sigma \otimes (R_P a)$$

Now μ vanishes on the elements of $(R_P\varphi) \otimes \ker \varphi \cong R_P \otimes \ker \varphi_i \cong \ker \varphi$, but μ is an isomorphism, so $\ker \varphi_i = 0$, demonstrating that φ_i is injective, and therefore an isomorphism.

Finally, let $\iota : (a_1, \ldots, a_n) \hookrightarrow I$ be the inclusion map. Pick any P. Since $I_P \cong R_P$ via a map taking some a_i to a generator of R_P, that a_i generates I_P, so ι_P is surjective. Since P was arbitrary, ι is surjective. □

Line bundles, and therefore invertible modules, are critical in the theory of schemes, in part because there is a tight connection between line bundles and maps into projective space (alternatively, one could argue that projective space is important in algebraic geometry because of its connection to line bundles). We will see a key example in Sam's presentation on Kahler differentials. On a calculational level, we see from this that the isomorphism classes of invertible modules of a Noetherian ring R form an abelian group, with identity R.

Definition 2.4. Let R be a ring, with S its set of non-zero divisors. We define $\mathcal{K}(R)$, the *total fraction ring* of R, to be $S^{-1}R$.

Definition 2.5. We call a finitely generated submodule of $\mathcal{K}(R)$ a *fractional ideal* of R.

The following theorem, which we will not prove, suggests another way of thinking of invertible modules: they have something to do with restrictions on zeros and poles.

Theorem 2.6. *Let R be a Noetherian ring and I be an invertible module. Then there is some fractional ideal Q so that $I \cong Q$.*

A quick and dirty estimate of the correct intuition is that the 'numerator part' corresponds to requiring that functions have zeroes of specified orders on codimension one subsets, while the 'denominator part' corresponds to allowing functions to have poles of specified orders on codimension one subsets.