Cluster Algebras

Jungho Daniel Choi

Presented: 29 April 2019

1 Goal & Overview

Cluster algebras were first introduced by Sergey Fomin and Andrei Zelevinsky in a series of papers published in the early 2000s, to further develop their work on Lie theory. Despite this specific origin, connections between cluster algebras and diverse areas of mathematics have emerged, as various objects have been shown to possess a cluster algebra structure. These areas range from tropical geometry to Teichmüller theory to mathematical physics. This has led mathematicians to begin treating cluster algebra structures as interesting objects of study in and of themselves.

Generally, a cluster algebra of rank n is a commutative subalgebra (and therefore, a subring) of an ambient field of rational functions in n variables. Having said this, given how deep this topic is, we will not construct a cluster algebra in its full generality here; we will restrict this brief write-up to defining a skew-symmetric cluster algebra of geometric type (of rank n). In doing so, rather than going down the rabbit hole, we shall merely take a peek at the structures within. Most commutative rings are typically defined by presenting a complete set of generators and relations; however, cluster algebras are better defined by defining an initial 'seed', each with a set of generators called 'cluster variables', then providing an operation by which to create new cluster variables. We demonstrate this below.

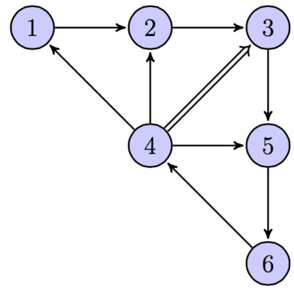
2 Skew-symmetric Cluster Algebras of Geometric Type

Remark 1. Throughout these notes, let $m \geq n$ be positive integers.[1] Also let $\mathcal{F}$ be an ambient field of rational functions in n independent variables over $\mathbb{Q}(x_{n+1}, \ldots, x_m)$, i.e. $\mathcal{F} = \mathbb{Q}(x_{n+1}, \ldots, x_m)(x_1, \ldots, x_n)$.

Definition 1. Quiver
A *quiver* Q is a directed graph given by a set of vertices and arrows, with the possibility of having multiple arrows between vertices. It is represented by an $m \times m$, integer-valued matrix that is skew-symmetric,[2] where Q_{ij} denotes the number of arrows going from vertex $i \to j$.

[1] This will be used to define a cluster algebra of rank n, by using a 'quiver' with m vertices.
[2] A skew-symmetric matrix is one where $Q_{ij} = -Q_{ji}$.

Example 1. Let us examine the following quiver:



It is represented by the matrix $Q = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$.

Definition 2. Extended cluster of seed
Given $\mathcal{F}$ the ambient field defined above, let $(x_1, \ldots, x_m)$ be a free generating set[3] for $\mathcal{F}$. Then, $\mathbf{x} = (x_1, \ldots, x_m)$ is the *extended cluster* of a seed. Of the variables in the extended cluster, we define $\{x_1, \ldots, x_n\}$ to be the *cluster variables*, which are the generators, and define $\{x_{n+1}, \ldots, x_m\} = c$ to be the *frozen/coefficient variables*.

Definition 3. Labelled seed in $\mathcal{F}$
Let $\mathbf{x}$ be an extended cluster of a seed in $\mathcal{F}$. Now, let Q be a quiver on vertices $1, \ldots, m$, with x_i in $\mathbf{x}$ associated with vertex i in the quiver. Then, there are *mutable vertices* $1, \ldots, n$ and *frozen vertices* $n+1, \ldots, m$. Then, $(\mathbf{x}, Q)$ is a *labelled seed in* $\mathcal{F}$.

Having defined a labelled seed with cluster variable generators, we now define the operation of 'mutation' to be able to create other seeds from an initial one.

Definition 4. Mutation over vertex k
Let $k \in \{1, \ldots, n\}$.[4] A *mutation over a vertex* k, denoted μ_k, on an initial seed $(\mathbf{x}, Q)$ is an operation that will map the initial seed to a new seed $(\mathbf{x}', Q')$ by a set of rules to produce a new quiver and a new extended cluster. The original quiver will be changed by the rules:

- For all paths $i \to k \to j$, 'complete the triangle' by drawing a new path $i \to j$.
- Reverse all arrows going in or out of vertex k.
- Remove any 2-cycles that have formed.

[3]For this tuple to be a free generating set, it must be that $x_1, \ldots, x_m$ are algebraically independent and that $\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_m)$.
[4]This condition restricts mutations to occur only over mutable vertices, rather than the frozen ones.

This will produce a new quiver that will be represented by a new matrix such that

$$Q'_{ij} = \begin{cases} -Q_{ij} & k \in \{i, j\} \\ Q_{ij} + Q_{ik}Q_{kj} & Q_{ik} > 0, Q_{kj} > 0 \\ Q_{ij} - Q_{ik}Q_{kj} & Q_{ik} < 0, Q_{kj} < 0 \\ Q_{ij} & \text{otherwise} \end{cases}.$$

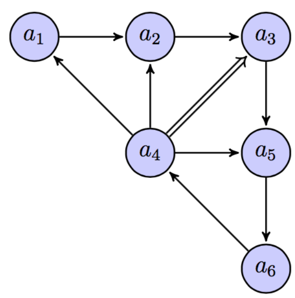
The original extended cluster will be changed by the following formula[5]:

$$\forall\, x_l \text{ in } \mathbf{x},\ x'_l = \mu_k(x_l) = \begin{cases} x_l & k \neq l \\ \frac{1}{x_k}\left[\prod_{i\to k} a_i + \prod_{k\to j} a_j\right] & k = l \end{cases}.$$

Remark 2. By the way that the mutation operation is defined, arrows between frozen vertices in a quiver affect neither the quiver mutation nor the extended cluster mutation.

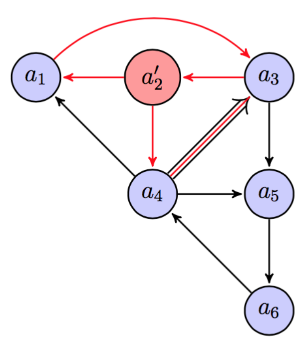
Proposition 1. Mutation on a vertex k is an involution, i.e. $\mu_k^2 = \mathrm{Id}$.

Example 2. We illustrate the process of mutation via an example. We take the quiver from the previous example and promote the vertices with a set of cluster variables $\mathbf{a} = (a_1, \ldots, a_6)$. As mutations are not affected by frozen variables, we assume there are none, and that all variables are cluster variables, for the sake of simplicity. The seed $S = (\mathbf{a}, Q)$ is the following:



.

We perform a mutation over the cluster variable a_2. This then yields the new seed $S' = (\mathbf{a}', Q')$, which can be seen as the following:



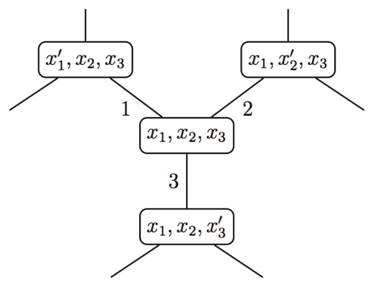
.

[5]Here, the notation $i \to k$ and $k \to j$ indicates that the product should be taken over all the arrows that are incoming to, or outgoing from k, respectively. Multiple arrows should be counted multiple times.

Here, $a_2' = \frac{1}{a_2}(a_1 a_4 + a_3)$, and $Q' = \begin{pmatrix} 0 & -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & -3 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$.

Definition 5. Cluster pattern
Let $\mathbb{T}_n$ be the infinite n-ary tree[6] with labelled edges $1, \ldots, n$ such that each of the n edges adjacent to a vertex will have a distinct label. A *cluster pattern* is the assignment of a labelled seed $\Sigma_t = (\mathbf{x}_t, Q_t)$ to every vertex $t \in \mathbb{T}_n$ such that the seeds assigned to adjacent vertices t, t' can be obtained from each other using a mutation μ_k.

Example 3. We provide an example of a cluster pattern for the case $n = 3$.



We may finally define a (skew-symmetric) cluster algebra (of geometric type) of rank n.

Definition 6. Cluster algebra of rank n
Let $\chi = \bigcup\limits_{t \in \mathbb{T}_n} \mathbf{x}_t = \{x_{i,t} \mid t \in \mathbb{T}_n, 1 \leq i \leq n\}$. Then, a cluster algebra $\mathcal{A}$ of rank n is a $\mathbb{Z}[c]$-subalgebra[7] of the ambient field $\mathcal{F}$ generated by all cluster variables, or $\mathbb{Z}[c][\chi]$. It is of rank n as each cluster will have n cluster variables.

Remark 3. Because the cluster algebra is independent of the choice of initial seed, we may denote $\mathcal{A} = \mathcal{A}(\mathbf{x}, Q)$, such that $(\mathbf{x}, Q)$ is any seed in the cluster pattern. An intuitive way to think of a cluster algebra is to think of it as the collection of seeds that can be obtained from an initial seed by mutating on it with every possible sequence of mutations.

Remark 4. Though cluster algebras have found many applications in mathematics, one we can briefly describe is that the combinatorics of the set of triangulations of any polygon has been found to have a cluster algebra structure and can be associated with what is known as a 'type A' cluster algebra. Specifically, the set of all triangulations of a d-gon is associated with a $\mathcal{A}_{d-3}$ cluster algebra (a type A cluster algebra of rank $d-3$).

[6] i.e. a connected acyclic simple n-regular graph
[7] Recall $c = \{x_{n+1}, \ldots, x_m\}$.

References

[1] Melody Chan, *A Survey of Cluster Algebras*.

[2] Joey Randich, *I Like My Algebras the Same as My Cereal: with Clusters*, 2018.

[3] Lauren K. Williams, *Cluster Algebras: An Introduction*, 2014.