Exponential Generating Functions

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1 Introduction

We are often interested in counting things. For example, count the number of ways to create a string of length n with the letters a,b, and c, such that there are an even number of a's and an odd number of b's. The solution to a problem like this is a sequence, where the n^{th} number in the sequence corresponds to the answer for strings of length n. Exponential generating functions provide a way to encode the sequence as the coefficients of a power series. This encoding turns out to be useful in a variety of ways.

Definition 1. *A class of permutations, A, is an association to each finite set X a set of permutations on X, A_X, such that $\#X = \#Y \implies \#A_X = \#A_Y$*

Definition 2. *Suppose a_n gives $\#A_{[n]}$ for some class A. If*

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

Then f is the exponential generating function for A. It is also said to be the exponential generating function for the sequence a_i.

2 Working with Generating Functions

Proposition 1. *Let a_n, b_n be two sequences with exponential generating functions $A(x), B(x)$. Then $A(x) + B(x)$ is the exponential generating function for the element-wise sum of the two sequences. $A(x) \cdot B(x)$ is the exponential generating function for the sequence*

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

Suppose that a_n is the sequence for a class of permutations A. Similarly for b_n and a class B. Consider the number of ways to divide n labeled objects into two groups, assign to the first group a permutation in A, and assign to the second group a permutation in B. If the first group is of size i, then the number of ways is $\binom{n}{i} a_i b_{n-i}$. Thus the total number of ways is $\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$. Look familiar? This number is equal to c_n.

Example 1. How many ways are there to make a string of length n using the letters a, b, and c, such that there are an odd number of a's and an even number of b's? Define 3 classes of permutations:

- A is the class of all identity permutations that are defined on a finite set with odd cardinality. This has sequence 0, 1, 0, 1, ...
- B is the class of all identity permutations that are defined on a finite set with odd cardinality. This has sequence 1, 0, 1, 0, ...
- C is the class of all identity permutations of any size. This has sequence 1, 1, 1, 1, ...

These classes have corresponding generating functions:

$$A : \left(\sum_{\text{n odd}} \frac{x^n}{n!} \right), B : \left(\sum_{\text{n even}} \frac{x^n}{n!} \right), C : \left(\sum_{n} \frac{x^n}{n!} \right)$$

Going back to our problem, think of each position in the string as an object. We want the number of ways to divide the positions into 3 groups, such that the A group contains an odd number of positions, the B group contains an even number of positions, and the C group can contain any number. This is exactly the number of ways to divide $[n]$ into 3 subsets S_A, S_B, S_C and assign a permutation in A to S_A, a permutation in B to S_B, and a permutation in C to S_C. By our previous reasoning, the answer to this problem must have generating function equal to the product of the generating functions for A, B, and C. Using our knowledge of Maclaurin series, this product is equal to

$$\left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) e^x = \frac{e^{3x} - e^{-x}}{4}$$

This is the generating function for the sequence

$$e_n = \frac{3^n - (-1)^n}{4}$$

You can check that the first few terms are correct!

Suppose as before that a_n is the corresponding sequence for a class of permutations A, and that a_n has exponential generating function $A(x)$. How many ways are there to arrange n labeled objects into some number of groups, and choose a permutation from A for each group? The groups themselves are not labeled. If we count the only the ways with exactly two groups, the answer will have exponential generating function $A(x)^2/2$ by our previous result about the product of generating functions. We divide by 2 since the two groups are indistinguishable. Similarly, if we count only the arrangements into k groups, then the answer will have exponential generating function $A(x)^k/k!$. Thus the desired generating function is

$$\sum_{k=0}^{\infty} \frac{A(x)^k}{k!} = e^{A(x)}$$

Example 2. How many ways can n people be arranged into pairs? Let this number be b_n. Consider the class A of permutations consisting only of the identity permutation on two elements. We want the number of ways to divide n people into some number of groups such that each group is an unordered pair. In other words we want the number of ways to divide n people into some number of groups and choose a permutation from A for each group. If the group is not of size two, then it is not possible to make such a choice! The exponential generating function for A has only one term, $x^2/2$. Applying our previous argument, the desired exponential generating function is given by $e^{x^2/2}$. We can expand the Maclaurin series to get a closed form solution of

$$b_n = \frac{n!}{2^{n/2}(n/2)!}$$

Sources.

http://www.cs.princeton.edu/courses/archive/fall04/cos341/exponential_generating_functions4.pdf

https://www.whitman.edu/mathematics/cgt_online/book/section03.02.html

http://www.math.uwaterloo.ca/~dgwagner/co430II.pdf