# DEFINING HOPF ALGEBRAS
# MATH 2520: PROF. MELODY CHAN

JACKSON MARKEY

I just want to start by acknowledging the book ”Hopf Algebras in Combinatorics” by Grinberg and Reiner, whose first chapter I follow fairly closely in these notes. Also note that, as the title implies, these notes do little more than lay the groundwork to establish a definition of Hopf algebras. For more examples and results, see an appropriate book like that listed above.

Hopf algebras arise a lot in combinatorics, algebraic topology, algebraic geometry, and related fields. In order to build up to their definition, we first establish a categorical definition for an algebra. Rather than defining it as the coslice category over a certain ring, we directly define its structure using product and unit morphisms.

Henceforth, let $R$ be a commutative ring.

**Definition 1.** An $R$-algebra is the following information:

- an $R$-module $A$
- an $R$-linear operation $\nabla : A \otimes A \to A$
- an $R$-linear unit $u : R \to A$

such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\nabla \otimes id} & A \otimes A \\ \big\downarrow{\scriptstyle \mathrm{id} \otimes \nabla} & & \big\downarrow{\scriptstyle \nabla} \\ A \otimes A & \xrightarrow{\quad \nabla \quad} & A \end{array} \qquad \begin{array}{ccccc} A \otimes R & \xleftarrow[\cong]{} & A & \xrightarrow{\cong} & R \otimes A \\ \big\downarrow{\scriptstyle \mathrm{id} \otimes u} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle u \otimes \mathrm{id}} \\ A \otimes A & \xrightarrow{\nabla} & A & \xleftarrow[\nabla]{} & A \otimes A \end{array}$$

Note that this definition allows for noncommutative algebras, so in some sense it is more general than the definition we use typically in this class. Remember, though, that our base ring $R$ is assumed to be commutative.

Now, if we think of the multiplication in an algebra as ”combining two elements of $A$ into one”, then the idea of a coalgebra is ”taking one element of $A$ and splitting it into two.” The below definition of a coalgebra is completely dual to that above of an algebra, in that we just reverse all the arrows.

**Definition 2.** An $R$-coalgebra is the following information:

- an $R$-module $A$
- an $R$-linear operation $\Delta : A \to A \otimes A$
- an $R$-linear counit $\epsilon : A \to R$

*Date*: 4/24/2019.

such that the following diagrams commute:

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xleftarrow[\Delta \otimes id]{} & A \otimes A \\
{\scriptstyle \mathrm{id} \otimes \Delta} \uparrow & & \uparrow {\scriptstyle \Delta} \\
A \otimes A & \xleftarrow[\Delta]{} & A
\end{array}
\qquad
\begin{array}{ccccc}
A \otimes R & \xrightarrow{\cong} & A & \xleftarrow[\cong]{} & R \otimes A \\
{\scriptstyle \mathrm{id} \otimes \epsilon} \uparrow & & {\scriptstyle \mathrm{id}} \uparrow & & {\scriptstyle \epsilon \otimes \mathrm{id}} \uparrow \\
A \otimes A & \xleftarrow[\Delta]{} & A & \xrightarrow{\Delta} & A \otimes A
\end{array}
$$

The left diagram expresses the property of ”coassociativity”, while the right diagram defines how the counit interacts with comultiplication. As you might guess, there is a dual property called ”cocommutativity” which a coalgebra may or may not posess.

In order to more easily express comultiplication, it is common to use the following notation, called *Sweedler notation*.

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2$$

This makes sense because performing comultiplication on an element always results in something in the tensor product, whose elements can always be written as some finite sum of pure tensors. The index $(c)$ just serves to remind us where we got our tensor from via comultiplication.

At first, the counit seems a little weird, but some examples with hopefully give a better idea of what coalgebras look like.

**Example 1.** Given an $R$-module $V$, we have the *tensor algebra* $T(V) = \bigoplus_{n \in N} V^{\otimes n}$. We can define comultiplication on this module by simply defining it on $v \in V$ (think of the adjoint pair containing the the tensor algebra functor). We define

$$\Delta(v) = v \otimes 1 + 1 \otimes v \qquad \epsilon(v) = \begin{cases} v & v \in V^{\otimes 0} \cong R \\ 0 & v \in V^{\otimes n}, n > 0 \end{cases}$$

and extend both linearly to all of $T(V)$. We can check coassociativity on our generating elements as follows

$$\begin{aligned}
\Delta \otimes \mathrm{id}(\Delta(v)) &= \Delta \otimes \mathrm{id}(v \otimes 1 + 1 \otimes v) \\
&= \Delta \otimes \mathrm{id}(v \otimes 1) + \Delta \otimes \mathrm{id}(1 \otimes v) \\
&= [(v \otimes 1 + 1 \otimes v) \otimes 1] + [(1 \otimes 1) \otimes v] \\
&= v \otimes 1 \otimes 1 + 1 \otimes v \otimes 1 + 1 \otimes 1 \otimes v \\
&= \mathrm{id} \otimes \Delta(\Delta(v))
\end{aligned}$$

and from this, we get that coassociativity holds on the whole space.

In fact, in this example we have a formula for calculating the coproduct of pure tensors that is very useful.

$$\Delta(v_1 \dots v_n) = \sum_{I \subset \{1,\dots,n\}} v_I \otimes v_{\{1,\dots,n\}-I}$$

where $I$ ranges over all increasing multi-index subsets of $I$, or all subwords of the word $123 \dots n$.

**Example 2.** In a similar manner, the *symmetric algebra* on an $R$-module $V$ is a $R$-coalgebra. Recall our construction of the symmetric algebra as a quotient of the form

$$S(V) = T(V)/\langle vw - wv : v, w \in V \rangle$$

One can define the concept of a *two-sided coideal* of a coalgebra, which allows one to take the quotient of a coalgebra and obtain a quotient coalgebra.[1] It turns out that the ideal generated by $vw - wv$ is indeed a coideal, and so directly from the tensor algebra we inherit a coalgebra structure.

**Example 3.** For a group $G$, we can define the *group coalgebra* $R[G]$ to be the set of formal sums of group elements. That is,

$$R[G] = \left\{ \sum_{g \in G} cg : c \in R \right\}$$

with comultiplication defined on the group elements as

$$\Delta(g) := g \otimes g$$

and counit defined by

$$\epsilon(g) = 1 \quad \forall g \in G$$

Thus, the counit evaluated on an element is just the sum of its different coefficients.

$$\epsilon\left( \sum_{g \in G} c_g g \right) = \sum_{g \in G} c_g \in k$$

One can check that with these definitions, $R[G]$ does indeed form a coalgebra.

Elements of any coalgebra whose comultiplication follows the pattern of the above examples are considered somewhat special.

**Definition 3.** An element $x \in C$ of a coalgebra is called

- primitive if $\Delta(x) = 1 \otimes x + x \otimes 1$
- group-like if both $\Delta(x) = g \otimes g$ and $\epsilon(g) = 1$

In addition to this new structure of a coalgebra that we just defined, each of our examples already has an algebra structure which we are already familiar with from earlier in this course. In fact, these two structures play well in a way that is formalized by the definition of a bialgebra.

**Definition 4.** An $R$-bialgebra is a $R$-module $A$ which is both an algebra $(A, \nabla, u)$ and a coalgebra $(A, \Delta, \epsilon)$ such that the maps $\Delta$ and $\epsilon$ are algebra morphisms.

[1] A submodule $J \subset C$ of a coalgebra is called a *two-sided coideal* if $\Delta(J) \subset J \otimes C + C \otimes J$ and $\epsilon(J) = 0$. See if you can work out a universal property for the quotient algebra by a coideal.

This last requirement that $\Delta$ and $\epsilon$ be algebra morphisms is just one way of requiring that the algebra and coalgebra stucture "play nice" with each other.

It turns out that a bialgebra $A$ naturally has an associative algebra structure on $\mathrm{Hom}(A, A)$, where the multiplication is given by convolution of maps. That is, given two $f, g \in \mathrm{Hom}(A, A)$ we define

$$f * g(a) := \sum_{(a)} f(a_1)g(a_2)$$

where the above sum is an example of Sweedler notation. We can also think of convolution as the composition of the following diagram.

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\nabla} A$$

**Claim 1.** The triple $(\mathrm{Hom}(A, A), *, 1 \mapsto u \circ \epsilon)$ form an algebra.

Since we already know that these functions form an $R$-module, all that remains is to check associativity of convolution and the existence of a unit.

- Let $f, g, h \in \mathrm{Hom}(A, A)$. Then for any $a \in A$,

$$\begin{aligned}((f * g) * h)(a) &= \sum_{(a)} f * g(a_1)h(a_2) \\ &= \sum_{(a)} \left( \sum_{(a_1)} f(a_{1,1})g(a_{1,2}) \right) h(a_2) \\ &= (\nabla) \circ (\nabla \otimes \mathrm{id}) \circ (f \otimes g \otimes h) \circ (\Delta \otimes \mathrm{id}) \circ (\Delta)(a)\end{aligned}$$

  but by associativity of multiplication and comultiplication in $A$, this is the same as splitting the other way and getting

$$\begin{aligned}&= (\nabla) \circ (\mathrm{id} \otimes \nabla) \circ (f \otimes g \otimes h) \circ (\mathrm{id} \otimes \Delta) \circ (\Delta)(a) \\ &= \sum_{(a)} f(a_1) \left( \sum_{(a_2)} g(a_{2,1})h(a_{1,2}) \right) \\ &= \sum_{(a)} f(a_1) g * h(a_2) \\ &= f * (g * h)(a)\end{aligned}$$

  so indeed the convolution is associative.
- It turns out that the map $u \circ \epsilon \in \mathrm{Hom}(A, A)$ acts as the identity element in this convolution algebra. The easiest way to see this is by stacking our unit diagram

on top of our counit diagram, and replacing the id in our unit diagram with $f$.

$$
\begin{array}{ccccc}
A \otimes A & \xrightarrow{\nabla} & A & \xleftarrow[\nabla]{} & A \otimes A \\
{\scriptstyle f\otimes u}\uparrow & & {\scriptstyle f}\uparrow & & {\scriptstyle u\otimes f}\uparrow \\
A \otimes R & \longleftrightarrow & A & \longleftrightarrow & R \otimes A \\
{\scriptstyle \mathrm{id}\otimes\epsilon}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \epsilon\otimes\mathrm{id}}\uparrow \\
A \otimes A & \xleftarrow[\Delta]{} & A & \xrightarrow{\Delta} & A \otimes A
\end{array}
$$

Follow the arrows from the bottom center to the top center to see that it acts as a unit in this algebra.

Finally we may define the titular object of this write-up.

**Definition 5.** A Hopf algebra over $R$ is an $R$-bialgebra along with a map $S \in \mathrm{Hom}(A, A)$, called the antipode, that is a two-sided inverse for $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ under convolution. In other words, the following diagram commutes.

$$
\begin{array}{ccccccc}
 & \nearrow^{\Delta} & A \otimes A & \xrightarrow{S\otimes \mathrm{id}_A} & A \otimes A & \searrow^{\nabla} & \\
A & & \xrightarrow{\epsilon} & R & \xrightarrow{u} & & A \\
 & \searrow_{\Delta} & A \otimes A & \xrightarrow{\mathrm{id}_A\otimes S} & A \otimes A & \nearrow_{\nabla} &
\end{array}
$$

Each of our previous examples can be given the structure of a bialgebra and, on top of that, a Hopf algebra[2]. I highly recommend taking a look at "Hopf Algebras in Combinatorics" by Grinberg and Reiner if you want to see more nice properties[3] of Hopf algebras or more in-depth examples. Now that you know the basic definition, you can go out and see how it applies to what you care about, whether it's affine group schemes[4] from algebraic geometry, cohomologies over Lie groups from algebraic topology, or the ring of symmetric functions from combinatorics.

[2]For $T(V)$, the antipode sends each generator $v \mapsto -v$, and for $R[G]$, the antipode sends each group element $g \mapsto g^{-1}$. From these examples, one may see why the antipode is sometimes also called the "anti-inverse", and more generally the antipode is one example of an "antihomomorphism"

[3]For example, under certain finiteness condition on the underlying module, the dual of a Hopf algebra is canonically a Hopf algebra.

[4]which are in correspondance with our group algebra $R[G]$