Symmetric Functions : Symmetric Functions

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This closely follows Chapter 7 of Stanley's *Enumerative Combinatorics*.

1 Introduction

Let $f(x_1, x_2, \ldots, x_n) \in R[x_1, x_2, \ldots, x_n]$ be a polynomial with coefficients in some ring R. We say f is a *symmetric polynomial* if for all permutations $\sigma \in S_n$, $f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$.

Symmetric polynomials are an important part of mathematics: they come up in Galois theory, since the coefficients of a polynomial are a symmetric polynomial evaluated at the roots of the polynomial, for example.

In that case, the symmetric polynomials all have the same number of variables. However, for all $n \geq 2$, if $e_1(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i$, $e_2(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i < j \leq n} x_i x_j$, and $p_2(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$, we have the identity

$$(e_1(x_1, x_2, \ldots, x_n))^2 = p_2(x_1, x_2, \ldots, x_n) + e_2(x_1, x_2, \ldots, x_n).$$

This suggests that there is something to these polynomials that doesn't depend too much on the number of variables, providing some motivation for considering the ring of formal power series in countably many variables, $R[[x_1, x_2, \ldots]]$.

We let the *set of homogeneous symmetric functions of degree* n, $\Lambda_R^n \subset R[[x_1, x_2, \ldots]]$, be the set consisting of all formal power series f such that the degree of each monomial of f is n and f is invariant under any permutation of the natural numbers, that is, $f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$.

As a matter of notation, given a (possibly infinite) tuple of integers which add up to n, $\alpha = (\alpha_1, \alpha_2, \ldots)$, we write x^α to mean $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$.

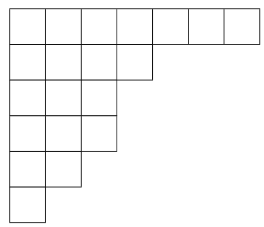
Then, we define the *ring of symmetric functions* as $\Lambda_R = \bigoplus_{n=1}^{\infty} \Lambda_R^n$. Note that this implies that symmetric functions have bounded degree. In addition, Λ_R has a R-algebra structure, since symmetric functions can be multiplied. From now on, we'll be working over $\mathbb{Q}$, so we'll just write Λ for $\Lambda_{\mathbb{Q}}$. In this case, Λ is a vector space.

2 Monomial Symmetric Functions

One important aspect of symmetric function theory is finding bases for Λ. To do this, we have to talk a bit about partitions.

We say a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a *partition* of $n \in \mathbb{N}$ if $\lambda_i \in \mathbb{N}$ for all i and $\sum_i \lambda_i = n$. We'll often write a partition omitting the parentheses and commas, like $\lambda = \lambda_1\lambda_2 \ldots \lambda_k$. We abbreviate the sentence "λ partitions n" using the symbols $\lambda \vdash n$.

Each partition λ has a *Young tableau* associated to it, which is perhaps best illustrated with an example. Given the partition 743321 of 20, the associated Young tableau has 7 boxes in the top row, 4 in the second, 3 in the third, and so on.



Partitions are important to the study of Λ because bases of Λ are often indexed by the set of all partitions of all natural numbers.

One basis of Λ indexed by partitions is the set of *monomial symmetric functions*. We define $m_\lambda = \sum_\alpha x^\alpha$, where $\lambda \vdash n$ for some n, and α ranges over all distinct permutations of λ.

Let $\mathscr{M} = \{m_\lambda\}$ be the set of monomial symmetric functions. To see that $\mathscr{M}$ spans Λ as a $\mathbb{Q}$-vector space, note that if a homogeneous symmetric function of degree n contains a term $c_\alpha x^\alpha$, it must also contain $c_\alpha x^\beta$, where β is a permutation of α. Additionally, for a given n, none of the $\{m_\lambda : \lambda \vdash n\}$ is a linear combination of any others, since their monomials are all different.

Finally, recall that $\Lambda = \mathbb{Q} \oplus \Lambda^1 \oplus \Lambda^2 \oplus \cdots$, so every symmetric function can be written uniquely as a linear combination of elements of Λ^n, which can also be written uniquely as a linear combination of elements of $\{m_\lambda : \lambda \vdash n\}$. Therefore, $\mathscr{M}$ is a basis for Λ.

3 Elementary, My Dear Watson

Now we're in a position to define the *elementary symmetric functions*. Given some $k \in \mathbb{N}$, let $e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$. This should align with the idea of elementary symmetric polynomials. Now, given a partition $\lambda = \lambda_1 \cdots \lambda_k$, define $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$. Let $\mathscr{E} = \{e_\lambda\}$ be the set of elementary symmetric functions.

It is a fact known as the fundamental theorem of symmetric functions that $\mathscr{E}$ is a basis for Λ, but we will not prove this here, since it requires us to talk about orderings on partitions.

We can, however, say something about the coefficients of the m_λ in the expansion of an element of $\mathscr{E}$ in terms of $\mathscr{M}$.

If $\lambda \vdash n$, and $e_\lambda = \sum_{\mu \vdash n} c_{\lambda\mu} m_\mu$, where $c_{\lambda\mu} \in \mathbb{Q}$, then $c_{\lambda\mu}$ is equal to the number of infinite matrices whose only entries are 0's and finitely 1's such that the sequence of row sums is λ and the sequence of column sums is α, where α is any permutation of μ.

To prove this, consider the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots \\ x_1 & x_2 & x_3 & \cdots \\ x_1 & x_2 & \ddots & \\ \vdots & \vdots & & \end{pmatrix}.$$

Recall that $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$. Thus, a term of e_λ is the product of λ_i choices from row i for each $i \le k$. If x^α is the term, then we have chosen α_j copies of x_j from column j. Now, if we mark the entries we chose with a 1, leaving all the other to be 0, we obtain a matrix whose sequence of row sums is λ, since there are λ_i 1's in row i, and whose sequence of column sums is α, since there are α_j 1's in column j.

Given such a matrix, we can use it to choose variables to create a term of e_λ. Therefore, the number of appearances of a term x^α, where α is a permutation of μ, is the number of such matrices. Since the number of appearances of a term x^α is the coefficient of m_μ, we have what we wanted.

Hopefully this example gives the sense that symmetric functions, and especially the coefficients of transformations between bases, can have nice combinatorial properties.