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Definition 1. Simplicial Complex Δ on $V = \{x_1, ..., x_n\}$ is a collection of subsets of V such that if $F \in \Delta$ and $G \subseteq \Delta$, then $G \in \Delta$. The members of Δ are called *faces* or *simplices*, and a face of Δ which is not properly contained in another face of Δ is called a *facet*. If all the facets of Δ have the same cardinality, then Δ is called pure.

Definition 2. The *dimension* of a face $A \in \Delta$ is one less than the cardinality of A. *dimension* of Δ is also defined as the maximum dimension of faces of Δ, in other words, $dim\Delta = Max_{F\in\Delta} dim(F) = d - 1$, where d is the cardinality of the facet of Δ with the largest cardinality.
Also, f-vector of Δ is vector $f(\Delta) = (f_{-1}, f_0, f_1, ..., f_{d-1})$, where f_i is the number of faces of Δ with dimension i. So $f_{-1} = 1$ when Δ is not void, and $f_0 = n$.

Example 1. $n = 5$, Δ is all subsets of $\{1, 2, 3\}, \{2, 4\}, \{3, 4\}, \{5\}$. (See Figure 1.) The f-vector of Δ is $(1, 5, 5, 1)$.

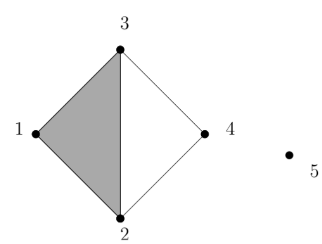

FIGURE 1. Simplicial Complex.

Example 2. Fix some l. If $l \leq k$, the convex hull of $k+1$ points in general position in $\mathbb{R}^l$ is a geometric k-simplex. In general, a collection Δ of geometric simplices in $\mathbb{R}^l$ is a geometric simplicial complex, if $\sigma \cap \epsilon$ is a geometric simplex in Δ for any $\sigma, \epsilon \in \Delta$. It is a basic fact from combinatorial topology that every simplicial complex has a geometric realization.

Definition 3. Let k be a field, and $R = k[x_1, x_2, ..., x_n]$. I_Δ is the ideal of R generated by the monomials $x_{i_1}x_{i_2}...x_{i_r}$, such that $\{x_{i_1}, x_{i_2}, ..., x_{i_r}\}$ is not a face of Δ. So I_Δ is a homogeneous ideal.

Example 3. In the example 1 above,
$I_\Delta = (x_1x_5, x_2x_5, x_3x_5, x_4x_5, x_1x_4, x_2x_3x_4)$.

Definition 4. Let F be a face of a simplicial complex Δ. P_F denotes the prime ideal of R generated by the indeterminates not in F. We call P_F the *face ideal* of F.

proposition 1. The primary decomposition of I_Δ is given by $I_\Delta = P_{F_1} \cap P_{F_2} \cap ... \cap P_{F_m}$, where $F_1, F_2, ..., F_m$ are facets of Δ. ($m = f_{d-1}$ if Δ is pure.)

Proof. Let J be any ideal generated by monomials of R, and a and b be relatively prime monomials. Then it is easily seen that $(ab, J) = (a, J) \cap (b, J)$. By using this equation successively we see that I is an intersection of ideals generated by subsets of $V = \{x_1, x_2, ..., x_n\}$. Now suppose $I_\Delta \subseteq (x_{i_1}, x_{i_2}, ..., x_{i_r})$. Now consider the set $\sigma = V \setminus \{x_{i_1}, x_{i_2}, ..., x_{i_r}\}$. Note

that $x_{i_1}x_{i_2}...x_{i_r}$ cannot be in I_Δ (since it is not in $(x_{i_1}, x_{i_2}, ..., x_{i_r})$), So $\sigma \in \Delta$, hence $(x_{i_1}, x_{i_2}, ..., x_{i_r}) = P_\sigma$.
Conversely, if $F \in \Delta$, then $I_\Delta \in P_F$. Because any generator monomials of I_Δ are in P_F. To see it is true, Consider any monomial $(x_{i_1}, x_{i_2}, ..., x_{i_r}) \in I_\Delta$ such that $(x_{i_1}, x_{i_2}, ..., x_{i_r}) \notin P_F$, so no x_{i_j} is in $V \setminus F$, hence $\{x_{i_1}, x_{i_2}, ..., x_{i_r}\} \subseteq F$, means that $\{x_{i_1}, x_{i_2}, ..., x_{i_r}\} \in \Delta$.
Therefore $I_\Delta = \cap_{F\in\Delta} P_F$. And since $F \subseteq G$ implies $P_G | P_F$, we have $I_\Delta = \cap_{F facet\ of \Delta P_F}$. □

Example 4. In the example 1 above,

$$I_\Delta = (x_1x_5, x_2x_5, x_3x_5, x_4x_5, x_1x_4, x_2x_3x_4) = (x_4, x_5) \cap (x_1, x_2, x_5) \cap (x_1, x_3, x_5), (x_1, x_2, x_3, x_4)$$

Definition 5. The *Stanley-Reisner Ring* associated to Δ is $A_\Delta = k[x_1, ..., x_n]/I_\Delta$. Since I_Δ is homogeneous ideal, there is grading of $A_\Delta : (A_\Delta)_i = R_i/(I_\Delta)_i$. This is also standard grading: $deg(x_i) = 1$ for all i.

proposition 2. For any simplicial complex Δ,

$$dim A_\Delta = dim \Delta + 1$$

Proof. By the last homework, Hilbert function of Δ is:

$$H(A_\Delta, m) = dim_k(A_\Delta)_m = \sum_{i=0}^{d-1} f_i \binom{m-1}{i}$$

Since dimension over k is an additive function, and the grading is standard, for sufficiently large m, Hilbert function of A_Δ is a polynomial in m of degree $dim A_\Delta - 1$. On the other hand, $\sum_{i=0}^{d-1} f_i \binom{m-1}{i}$ is a polynomial in m of degree $d-1$, so $d - 1 = dim A_\Delta - 1$. Hence $d = dim A_\Delta$. □

This is the second proof:

Proof. For Noetherian local ring (A,m), $dim(A)$ is the maximum number of algebrically independent elements of A over field k. It can be easily seen that $dim(A)$ is the maximum number of algebrically independent set of $x_{i_1}, x_{i_2}, ..., x_{i_j}$ over field k. For $r > d$, any r vertices $\{x_{i_1}, x_{i_2}, ..., x_{i_r}\}$ is not a face of Δ, so $x_{i_1}, x_{i_2}, ..., x_{i_r} = 0$ in A_Δ. Now consider $f(x_1, ..., x_r) = x_1x_2...x_r$, then f is not zero polynomial, but $f(x_{i_1}, x_{i_2}, ..., x_{i_r}) = 0$. So there is no r algebrically independent set of $x_{i_1}, x_{i_2}, ..., x_{i_j}$ over field k when $r > d$.
Now it suffices to show that there is indeed d algebrically independent set of $x_{j_1}, x_{j_2}, ..., x_{j_d}$. Consider facet $\{x_{i_1}, x_{i_2}, ..., x_{i_d}\}$ of Δ. Suppose $f(x_{i_1}, ..., x_{i_d}) = 0$. So $f(x_{i_1}, ..., x_{i_d}) \in I_\Delta$. So for each monomial of $f(x_{i_1}, ..., x_{i_d})$, there is a monomial $x_{s_1}x_{s_2}...x_{s_t}$ of I_Δ that divides it. Therefore, the set of indeterminates of each monomial of $f(x_{i_1}, ..., x_{i_d})$ contains $\{x_{s_1}, x_{s_2}, ..., x_{s_t}\}$, hence is not in Δ as well, which means that any monomial of $f(x_{i_1}, ..., x_{i_d})$ is zero in A_Δ. So $x_{i_1}, x_{i_2}, ..., x_{i_d}$ is an algebrically independent set over k. □

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