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Linear Algebraic Groups

Overview

A linear algebraic group is analogous to a topological group; it is an affine variety with a group structure, such that multiplication and the finding of inverses are morphisms of varieties. The general linear group $GL(n, K)$ can be considered a linear algebraic group, and indeed every linear algebraic group which is a variety of K is isomorphic to some subgroup of $GL(n, K)$.

(1)-(4) follow source [1]. The rest follow source [2].

Preliminaries

Affine n-space: Let K be a field. We call K^n affine n-space, and we write A^n. This is emphasizes the geometrical importance of the space.

Affine variety: The set of common zeros to a finite collection of polynomials called an affine variety. This is a curve, surface, or hyper-surface in affine n-space.

Zarisky topology: Consider A^n. One can set a topology on A^n in which the closed sets are exactly the affine varieties. This is the Zarisky topology.

We set the following relationships between ideals and varieties: Let I be an ideal of $k[x_1, ..., x_n]$. We set $V(I) = \{x \in A^n, f(x) = 0, \forall f \in I\}$, the set of common zeros of I. This is always an affine variety. By Hilbert's basis theorem, $k[x_1, ..., x_n]$ is Noetherian, that is every ideal I is finitely generated by some $\{f_i\}$, and the common zeros of I are exactly the common zeros of the generators. Since the set is finite, $V(I)$ is a variety. (But different ideals can generate the same variety.)

Let X be a variety of A^n. Set $I(X) = \{f \in k[x_1, ..., x_n], f(x) = 0, \forall x \in X\}$, the set of polynomials vanishing on X. Note that this is always an ideal, and furthermore it is always a radical ideal.

Hilbert's Nullstellensatz: $\sqrt{I} = I(V(I))$. This states that ideals having the same radical define the same ideal. In fact, it sets up an one-to-one, inclusion reversing correspondence between radical ideals and varieties.

Varieties

Affine algebra: Given an affine variety X of A^n, we let $k[X] = k[x_1, ..., x_n]/I(X)$. We call $k[X]$ the affine algebra of X, or the coordinate ring. We know $I(X)$ are exactly the polynomials vanishing on X. So $k[X]$ is exactly the ring of polynomials on X, in which polynomials are considered equal exactly when they are equal everywhere on X.

Note that by the Nullstellensatz the varieties on X, that is the varieties of A^n in X, correspond exactly to the radical ideals of $k[X]$.

Morphism of varieties: Before discussing what a morphism of algebraic groups looks like, we must determine what is a morphism of varieties. Let $X \subset A^n, Y \subset A^m$ be varieties. We want a function which is continuous with respect to the Zarisky topology. Since the closed sets are based on the zeros of polynomials, the "natural" morphisms are polynomial. Specifically, polynomials in $k[X]$ can be considered functions from X to A. The products for such functions are morphisms of varieties.

We define a morphism of varieties to be a function $f : X \to Y, x \mapsto (f_1(x), ..., f_m(x))$, where the $f_i \in k[X]$ That is, it is a function which is "polynomial in each coordinate."

Isomorphism of varieties: An isomorphism of varieties is a morphism $\varphi : X \to Y$ which is bijective, such that $\varphi^{-1} : Y \to X$ is also a morphism of varieties (i.e. φ has a polynomial inverse.)

(1) Morphisms of varieties are continuous

Proof: Let $\varphi : X \to Y$ be a morphism of varieties. We must show the inverse of every closed set is closed, that is the inverse of every variety is a variety.

Let $\{f_i\}$ be a set of polynomials defining a variety $Z \subset Y$. We can consider them to be functions from Y to A. So the polynomials $f_i \circ \varphi$ are polynomials on X. We have:

$x \in \varphi^{-1}(Z) \iff \varphi(x) \in Z \iff f_i(\varphi(x)) = 0, \forall i \iff f_i \circ \varphi(x) = 0, \forall i$

So the set $\varphi^{-1}(Z)$ is a variety, the variety defined by the $f_i \circ \varphi$. $\square$

Comorphisms: Let $\varphi : X \to Y$ be a morphism of varieties. We define the comorphism $\varphi^* : k[Y] \to k[X] : f \to f \circ \varphi$. It is a morphism of k-algebras. Indeed the * defines a contravariant functor from the category of varieties to the category of affine k-algebras, though we shall not go into this completely. Also, note that $\varphi^* = \psi^* \implies \varphi = \psi$.

(2) Functoral behavior of * on morphisms:

(i)$1_X^* = 1_{k[X]}$

(ii)Let $\varphi : X \to Y, \psi : Y \to Z$ be morphisms. $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$

(iii)Let $\varphi : X \to Y$ be a morphism. $\varphi^* : k[Y] \to k[X]$ is a k-algebra morphism.

Proof:

(i) $1^*(f) = f \circ 1 = f$

(ii) $(\varphi \circ \psi)^*(f) = f \circ \varphi \circ \psi = \psi^*(f \circ \varphi) = \psi^* \circ \varphi^*(f)$

(iii) Let $a \in k$.

$\varphi^*(af) = af \circ \varphi = a\varphi^*(f)$

$\varphi^*(f + g) = (f + g) \circ \varphi = f \circ \varphi + g \circ \varphi = \varphi^*(f) + \varphi^*(g)$ $\varphi^*(fg) = fg \circ \varphi = (f \circ \varphi)(g \circ \varphi) = \varphi^*(f)\varphi^*(g)\Box$

(3) Affine algebra homomorphisms correspond to unique morphisms of varieties:Let $XA^n, Y \subset A^m$ be varieties. Let $\Phi : k[Y] \to k[X]$ be a k-algebra homomorphism. There is a unique morphism $\varphi : X \to Y$ such that $\varphi^* = \Phi$.

Suppose that we have morphisms $\varphi, \psi : X \to Y$ such that $\varphi^* = \psi^*$. But this implies that $\varphi^*(\bar{y}_i) = \psi^*(\bar{y}_i)$ where the $\bar{y}_i$ are the reductions of the variables of y_i mod $I(Y)$. Thus $y_i \circ \varphi = y_i \circ \psi$. Let $\psi = (f_1, ..., f_m), \varphi = (g_1, ..., g_m)$. Then $y_i \circ \psi = f_i, y_i \circ \phi = g_i$. Thus $f_i = g_i$, and $\varphi = \psi$

Proof: We have $k[X] = k[x_1, ..., x_n]/I(X), k[Y] = k[y_1, ..., y_m]/I(Y)$. Let $\bar{\Phi} : k[y_1, ..., y_m] \to k[X] : g \mapsto \Phi(g + I(Y))$ be the lift of Φ. Note it is also a k-algebra homomorphism, as it is the composition of Ψ and the quotient map.

We choose representatives f_i such that $\bar{\Phi}(y_i) = \Phi(y_i + I(Y)) = f_i + I(X)$.

For all $g \in k[y_1, ..., y_m]$ we have:

$g(f_1, ..., f_m) + I(X) = g(\bar{\Phi}(y_1), ..., \bar{\Phi}(y_m)) = \bar{\Phi}(g(y_1, ..., y_m)) = \bar{\Phi}(g)$

We set $\varphi = (f_1, ..., f_m)$. We must check that $\varphi(X) \subset Y$. If so, it is a morphism of varieties. So we must show every polynomial in $k[Y]$ is zero at $\varphi(x)$ for all $x \in X$. Let $g \in k[Y]$. Note then that $g(f_1, ..., f_m) \in I(X)$.

$g(f_1 + I(X), ..., f_m + I(X)) = g(f_1, ..., f_m) + I(X) = \bar{\Phi}(g) = I(X)$

Also, we have $\varphi^* = \Phi$. Let $g \in k[y_1, ..., y_m]$:

$\varphi^*(g + I(Y)) = (g \circ \varphi) + I(X) = g(f_1, ..., f_m) + I(X) = \bar{\Phi}(g) = \Phi(g + I(Y))$

And so we have existence, and thus uniqueness. $\Box$

(4) Isomorphism of varieties depends on comorphism of algebras: Let $X \subset A^n, Y \subset A^m$ be varieties and let $\varphi : X \to Y$ be a morphism. φ is an isomorphism iff φ^* is an isomorphism of k-algebras, and then $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

Proof: Let $\varphi : X \to Y$ be an isomorphism. Then we have $\varphi^{-1} : Y \to X$ such that $\varphi \circ \varphi^{-1} = \varphi^{-1} \circ \varphi = 1_X$. But then by (2) $\varphi^* \circ (\varphi^{-1})^* = (\varphi^{-1})^* \circ \varphi^* = 1_{k[X]}$.

So φ^* is an isomorphism with $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

Let $\varphi^* : k[Y] \to k[X]$ be an isomorphism. Then there is a $(\varphi^*)^{-1} : k[X] \to k[Y]$ such that $\varphi^* \circ (\varphi^*)^{-1} = (\varphi^*)^{-1} \circ \varphi^* = 1_{k[X]}$. By (3) there is a morphism of varieties $\psi : Y \to X$ such that $(\varphi^*)^{-1} = \psi^*$.

$(\varphi \circ \psi)^* = \psi^* \circ \varphi^* = 1_{k[X]} = \varphi^* \circ \psi^* = (\psi \circ \varphi)^*$

By (2) and (3) this means $\psi \circ \varphi = \varphi \circ \psi = 1_X$. Thus φ is an isomorphism, and we have $(\varphi^*)^{-1} = (\varphi^{-1})^*$. $\square$

<u>Product of varieties:</u> Let $X \subset A^n, Y \subset A^m$ be varieties. We let the product $X \times Y$ be the usual subset of A^{n+m}, topologized by the Zariski topology. Note this is not the usual product topology.

Furthermore, this is actually the product in the category of varieties in the sense of category theory, but we shall not need this fact.

<u>Product of varieties is a variety</u>

<u>Proof:</u> Let $f_i : A^n \to k, g_i : A^m \to k$ be the finite sets of polynomials defining X and Y respectively, seen as functions. Then the set of products $f_i g_j : A^{n+m} \to k$ defines precisely the variety $X \times Y$. $\square$

<u>(5) $k[X \times Y] \cong k[X] \otimes k[Y]$</u>

<u>Proof:</u> Let $R = k[X], S = k[Y]$, $\sigma : R \otimes_k S \to k[X \times Y] : f \otimes g \mapsto f(T_1, ..., T_n)g(U_1, ..., U_m)$. We will show σ is an isomorphism of k-algebras.

Clearly any polynomial of $n + m$ indeterminates is a sum of products of polynomials in the first n and last m indeterminates. So σ is surjective.

Now we show injectivity. Let $f = \sum_{i=1}^{r} f_i \otimes g_i$ be a polynomial sent to 0, such that $f \neq 0$. Then $r = 1$: Not all g_i are 0, so let $y \in Y, g(y)$ are not all 0. Now we know $\sum f_i(x)g_i(y) = 0, \forall x \in X$. But this means $\sum g_i(y)f_i = 0$ in R, so the g_i are linearly dependent over k. So if $r > 1$, we could reduce it by 1. Hence $r = 1$.

Now by the above $f_1 = 0$, so $f = 0$.$\square$

Linear Algebraic Groups

<u>Linear Algebraic Group:</u> Let G be an affine variety (as opposed to a projective variety). We say G is a linear algebraic group, or an affine algebraic group, when there is a group structure on the points of G. We require that $\mu : G \times G \to G : (g, h) \mapsto gh$, the multipication of the group, is a morphism of

varieties, and that $i : G \to G : g \mapsto g^{-1}$, the inverse function, is a morphism of varieties.

Of course we also require there is an identity $1 \in G$ such that $1g = g1 = g, \forall g \in G$. We require inverses to have the usual property of inverses $gg^{-1} = g^{-1}g = 1$. And we require the associativity of multiplication $g(hi) = (gh)i, \forall g, h, i \in G$.

Note that not all subgroups of a linear algebraic group are sub-structures. The sub-structures of a linear algebraic group are the closed subgroups, that is subgroups which are closed under the Zariski topology (varieties).

It shall not be proved here, but $Im(\psi)$ is always a closed subgroup for any morphism of linear algebraic groups.

Morphism of Linear Algebraic Groups: Let $\varphi : G \to H$ be a function between linear algebraic groups. Then it is a morphism of linear algebraic groups if it is a group homomorphism and a morphism of varieties.

φ is an isomorphism of linear algbraic groups if it is an isomorphism of groups and of varieties.

General Linear Group:One vitally important algebraic group is the general linear group $GL(n, k)$ of invertible $n \times n$ matricies under multiplication. First we shall give it the structure of variety in A^{n^2+1}.

We identify the first n^2 coordinates with the corrdinate functions of the matrix T_{ij}. The n^2+1-th coordinate will be identified with $1/det(T)$. Since the matricies are invertible, the determinant is always invertible. Then $GL(n, k)$ can be identified with the variety of zeros of $(1/det(T)) * det(T_{ij}) - 1 = 0$. This is a polynomial, since $det(T_{ij})$ is a polynomial of the coordinates T_{ij}.

Now, this forms an algebraic group since given matricies $A, B \in GL(n, k)$, $\mu(A, B) = AB$ is polynomial in each coordinate. We have $AB_{ij} = \sum_k A_{ik}B_{kj}$, so the coordinates of the product AB are polynomials of the coordinates of A and B. Also, $1/(det(AB)) = (1/det(A))(1/det(B))$, so it is polynomial in the last coordinate as well.

The inverse $mathcalci(A) = A^{-1}$ is also polynomial in every coordinate. Note that the coordinates of A^{-1} are rearrangements of the corrdinates of A, perhaps with signs switched. These are all clearly polynomial. And the final coordinate $1/det(A^{-1}) = det(A_{ij})$ which is a polynomial of the coordinates of A. (This is a special case of a more general procedure to make a principle open set affine, that is a set $X_f = \{f(x) \neq 0\}$.

Further examples:

The multiplicative group G_m: This is essentially the set $k^* = k - \{0\}$ under multiplication. Since k is a field, we know k^* is a group. Note that we can consider this to be the group of 1×1 matricies, $G_m = GL(1,k)$. So we know it is truly a linear algebraic group over A^2.

The additive group G_a: This is the set k as a group under addition. This is clearly a variety with polynomial 0 over A. μ and i are clearly polynomial, with $\mu : G \times G \to G : (x,y) \mapsto x+y$, $i : G \to G : x \mapsto -x$.

Action of a Linear Algebraic Group: Let X be an affine variety and G be a linear algebraic group. An action of G on X is a morphism of varieties $\varphi : G \times X \to X : (g,x) \mapsto gx$ which follows the usual rules for group actions. That is, $1x = x, g(hx) = (gh)x, \forall x \in X, \forall g,h \in G$. We say that G acts morphically on X.

Translation of functions: Let $\varphi : G \times X \to X$ be a morphic action. Now consider the morphism of varieties $X \to X : y \mapsto yx, x \in G$. (It is a morphism since φ is). The comorphism is called a translation of functions: $\tau_x : k[X] \to k[X] : (\tau_x f)(y) = f(x^{-1}y)$.

Note that $\tau : G \to GL(k[X]) : x \mapsto \tau_x$,the group of automorphisms of $k[X]$, is a group homomorphism (in the usual sense):

$\tau_{gh}(f)(y) = f((gh)^{-1}y) = f(h^{-1}g^{-1}y) = \tau_h(f(g^{-1}y)) = \tau_g \circ \tau_h(f)(y)$

(6) Translation of functions lemma: Let $\varphi : G \times X \to X$ be an algebraic group action. Consider the comorphism $\varphi^* : k[X] \to k[G \times X] = k[G] \otimes k[X]$. Let F be a finite dimensional subspace of $k[X]$.

(a) There is a finite dimensional subspace E, such that $F \subset E$, which is fixed under translation of functions: $\tau_x E = E \forall x \in G$.

(b) F is fixed under translation of functions iff $\varphi^* F \subset k[G] \otimes_k F$.

Proof:

(a) We know F is finite dimensional, so we can select generators $F =< h_1, ..., h_n >$. Let h_j be one of these generators. We write, perhaps not uniquely $\varphi^* h_j = \sum f_i \otimes g_i \in k[G] \otimes k[X]$.

Let $x \in G, y \in X$. $\tau_x f(y) = f(x^{-1}y) = \sum f_i(x^{-1})g_i(y)$. Hence $\tau_x f = \sum f_i(x^{-1})g_i$. Thus, the g_i span a finite dimensional subspace of $k[X]$ which contains all possible translations of h_j. Now, the set of all the g_i for all h_j span a finite dimensional subspace E invariant under translation of functions. Also, note it contains $\tau_1 h_j = h_j$, all the generators of F. So $F \subset E$.

(b) Let $\varphi^* F \subset k[G] \otimes F$. Then in the proof of part (a), we can select g_i in F. So F itself is stable under translation of functions.

Let F be stable under translations. Extend the basis $\{f_i\}$ to a basis $\{f_i\} \cup \{g_j\}$ of $k[X]$. We let $\varphi^* f = \sum r_i f_i + \sum s_i g_j$. Then $\tau_x f = \sum r_i(x^{-1}) f_i + \sum s_j(x^{-1}) g_j$. This is in F, so the functions s_j vanish everywhere on G. Hence they are zero in $k[G]$. Thus, $\varphi^*F \subset k[G] \otimes F$. $\square$

Right translation of functions: By analogy, let G act on itself by right multiplication by x that is $y \mapsto yx$. The comorphism is the right translation of functions by x. $\rho_x f(y) = f(yx)$.

Note that $\rho : G \to GL(k[G]) : x \mapsto \rho_x$ is a group homomorphism.

Theorem: Let G be a linear algebraic group. Then G is isomorphic to a closed subgroup of some $GL(n,k)$:

Proof: Let G be a linear algebraic group. Choose generators f_i for $k[G]$. That is $k[G] = k[f_1, ..., f_n]$. Let $F = Span f_i$. Then by (6a) there is a finite dimensional subspace E such that $F \subset E$ and E is stable under translations of functions. We now let the f_i be a basis for E. This basis also generates $k[G]$.

Let $\varphi : G \times E \to E$ denote the natural restriction of the action of ρ. By (6b) we can write $\varphi^* f_i = \sum_j m_{ij} \otimes f_j$, where $m_{ij} \in k[G]$. Then $\rho_x f_i(y) = f_i(yx) = \sum m_{ij}(x) f_j(y)$. That is $\rho_x f_i = \sum m_{ij}(x) f_j$. Hence the matrix of ρ_x in the basis f_i is exactly $(m_{ij}(x))$. Let $\psi : G \to GL(n,k) : x \mapsto (m_{ij}(x))$. Note that since ρ is a group homomorphism, so is ψ. Furthermore, since it is polynomial in every coordinate, it is a morphism of varieties, and hence of algebraic groups.

ψ is injective: Let $x \in ker\psi$. Then ρ_x fixes all the f_i. So $\rho_x f_i(y) = f_i(yx) = f_i(y)$. So $f_i(x) = f_i(1)$. But this means $f(x) = f(1)$ for all $f \in k[G]$. So $x = 1$.

ψ is an isomorphism of varieties: Note that ψ^* sends the coordinate functions T_{ij} to the polynomials m_{ij}. But we have $f_i(x) = f_i(1x) = \sum m_{ij}(x) f_i(1)$. This shows the m_{ij} generate $k[G]$, so ψ^* is surjective. Also, ψ is bijective $\implies$ ψ^* is injective. So ψ^* is an isomorphism and by (4) we are done. $\square$

Bibliography

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