Another Proof of Clairaut’s Theorem

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Abstract. This note gives an alternate proof of Clairaut’s theorem—that the partial derivatives of a smooth function commute—using the Stone–Weierstrass theorem.

Most calculus students have probably encountered Clairaut’s theorem.

**Theorem.** Suppose that $f : [a, b] \times [c, d] \to \mathbb{R}$ has continuous second-order partial derivatives. Then $f_{xy} = f_{yx}$ on $(a, b) \times (c, d)$.

The proof found in many calculus textbooks (e.g., [2, p. A46]) is a reasonably straightforward application of the mean value theorem. More sophisticated techniques—Fubini’s theorem and Green’s theorem—can each be used to give easy proofs (for instance, [1, p. 61], exercise 3-28). The proof here relies on the density of two-variable polynomials in $C([a, b] \times [c, d])$. More precisely, we use the following version of the Stone–Weierstrass theorem.

**Theorem.** Let $g \in C([a, b] \times [c, d])$. There is a sequence $p_n(x, y)$ of two-variable polynomials such that $p_n \to g$ uniformly.

Applying the theorem to the continuous function $f_{xy}$ gives a sequence of polynomials $p_n$ such that

$$|p_n(x, y) - f_{xy}(x, y)| < \epsilon(n) \quad \text{for all} \quad (x, y) \in [a, b] \times [b, c]$$

where $\lim_{n \to \infty} \epsilon(n) = 0$.

Therefore, for any rectangle $R = [x_1, x_2] \times [y_1, y_2] \subset [a, b] \times [c, d]$,

$$\left| \iint_R p_n \, dx \, dy - \iint_R f_{xy} \, dx \, dy \right| < \epsilon(n) A(R), \quad (1)$$

where $A(R) = (x_2 - x_1)(y_2 - y_1)$ is the area of the rectangle $R$. Observe that

$$\iint_R f_{xy} \, dx \, dy = \iint_R f_{yx} \, dy \, dx ,$$

since both are equal to $f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1)$.

Since $p_n$ is a polynomial, it is a trivial computation to verify that

$$\iint_R p_n \, dx \, dy = \iint_R p_n \, dy \, dx$$

for each $n \in \mathbb{N}$ (this also follows from Fubini’s theorem, but even without assuming Fubini’s theorem, equality is straightforward since both integrals can be directly computed).

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Therefore, we also have
\[
\left| \iint_{R} p_n \, dy \, dx - \iint_{R} f_{yx} \, dy \, dx \right| < \epsilon(n) A(R).
\] (2)

Taking a limit as \( n \to \infty \), (2) becomes
\[
\iint_{R} f_{xy} - f_{yx} \, dy \, dx = 0.
\] (3)

Since \( f_{yx} - f_{xy} \) is continuous and (3) is true for all rectangles \( R \), \( f_{yx} - f_{xy} \) is identically zero, that is, \( f_{xy} = f_{yx} \).

As a side remark, the same approach proves the equality of iterated integrals in the Fubini theorem for continuous functions. To see this, given \( f \in C([a, b] \times [c, d]) \), take \( p_n \to f \), so
\[
\int_{a}^{b} \int_{c}^{d} p_n(s, t) \, ds \, dt \to \int_{a}^{b} \int_{c}^{d} f(s, t) \, ds \, dt
\]
\[
\int_{a}^{b} \int_{c}^{d} p_n(s, t) \, dt \, ds \to \int_{a}^{b} \int_{c}^{d} f(s, t) \, dt \, ds.
\]

As above, the two integrals on the left are equal for all \( n \), so by uniqueness of limits, the right hand sides are also equal.

REFERENCES


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