Statistical Mechanics and Combinatorics : Lecture V

Dimer Model

Last time we talked about

- The characteristic polynomial $P(z, w) = a + bz + cw$.
- The free energy $F$ for a dimer model on the honeycomb graph $H_n$

$$F = \frac{1}{(2\pi i)^2} \int_S \int_S \log(a + bz + cw) \frac{dz}{z} \frac{dw}{w}.$$  

- The probability for an edge of type $a$

$$\Pr(\text{an edge of type } a) = \frac{\theta_a}{\pi}$$

with the assumption that $a, b, c$ satisfy the triangle inequality and $\theta_a, \theta_b, \theta_c$ are the angles opposite sides $a, b, c$.

**Open Question:** Why $\Pr(\text{an edge of type } a)$ is such a simple geometric answer?

1 Inverse Kasteleyn matrix

Recall the probability $\Pr(e_1, \cdots, e_k)$ of a set of fixed edges $X = \{e_1, \cdots, e_k\}$ in a random dimer cover $M$ of honeycomb graph $H_n$ is determined by inverse Kasteleyn matrix

$$\Pr(e_1, \cdots, e_k \in M) = \det(K^{-1}(w_i, b_j))_{1 \leq i, j \leq k} \cdot \prod_{i=1}^{j} K(b_i, w_i).$$

As $n \to \infty$, the statement is identical so we shall define the infinite matrix $K^{-1}$ first.

**Definition 1.1.** The infinite matrix $K^{-1}$ is defined by

$$K^{-1}(w_0, 0, b_{x, y}) = \frac{1}{(2\pi i)^2} \int \frac{z^{-x} \omega^{-y}}{a + bz + cw} \frac{dz}{z} \frac{dw}{w},$$

where $w_{0,0}$ is the white vertex at the origin and $b_{x, y}$ corresponds to black vertex at $e_1 + x(e_3 - e_1) + y(e_1 - e_2)$, $e_1, e_2, e_3$ are unit vectors in the direction of the three cube roots of unity. $K^{-1}(w_{0,0}, b_{x, y})$ depends on the choice of white or black vertices.
Question: When \( b \) and \( w \) are far from each other, what can we say about \( K^{-1}(w, b) \)?

Note that the values of \( K^{-1}(w_0, b_{x,y}) \) are Fourier coefficients of \( \frac{1}{p(z,w)} = \frac{1}{a+bz+cw} \). When \( a, b, c \) satisfy the triangle inequality, \( \frac{1}{a+bz+cw} \) has two simple poles and they are complex conjugates.

Because of these poles, the coefficients decay linearly in \( |x| + |y| \), i.e.

\[
K^{-1}(w, b) = O\left(\frac{1}{|w-b|}\right).
\]

Question: Whether the probability of two distant edges is close to the product of their individual probabilities in a random dimer cover?

Definition 1.2. Let the correlation \( \text{Cor}(e_1, e_2) \) of two edges \( e_1, e_2 \) in a random dimer cover as

\[
\text{Cor}(e_1, e_2) = \Pr(e_1, e_2) - \Pr(e_1) \Pr(e_2).
\]

Correlation is an important measure of how quickly information is lost with distance in the covering.

Since

\[
\Pr(e_1, e_2) = a^2 \det\left(\begin{array}{cc} K^{-1}(b_1, w_1) & K^{-1}(b_1, w_2) \\ K^{-1}(b_2, w_1) & K^{-1}(b_2, w_2) \end{array}\right)
= a^2(K^{-1}(b_1, w_1)K^{-1}(b_2, w_2) - O\left(\frac{1}{d^2}\right))
= \Pr(e_1) \Pr(e_2) - O\left(\frac{1}{d^2}\right),
\]

the correlation of distant edges decays quadratically in the distance between them

\[
\text{Cor}(e_1, e_2) = O\left(\frac{1}{d^2}\right).
\]

2 Determinantal point process

Definition 2.1. A determinantal point process \( \mathcal{P} \) on \([n] = \{1, \cdots, n\}\) (we sometimes refer to elements of \([n] \) as elements) is a probability measure on \(\{0, 1\}^n\) with the property that for any subset \( S \subset [n] \), \( \exists \) a kernel \( M \) which is \( n \times n \) matrix such that

\[
\Pr(S) = \det(M^S_S).
\]

Here \( M^S_S \) is the submatrix of \( M \) keeping rows and columns indexed by \( S \).
Theorem 2.2. Let $M$ be a symmetric $n$ by $n$ matrix. $M$ is the kernel of a determinantal process $P$ for $[n]$ iff its eigenvalues are in $[0, 1]$.

Open Question: Classify all determinantal point processes for $[n]$.

Lemma 2.3. If $M$ is the kernel of a determinantal process $P$, for $x_1, \cdots, x_n \in \{0, 1\}^n$,

$$\Pr(x_1, \cdots, x_n) = |\det(\begin{pmatrix} 1 - x_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 - x_n \end{pmatrix} - M)|$$

Proposition 2.4.

1. If $P$ is a determinantal process with kernel $M$, then the expected number of particles is $Tr M$.

2. Let $P_j$ be the probability of $j$ particles. Then the probability generating function for the number of particles is

$$\sum_{j=0}^{n} P_j z^j = \det(I - M + zM) = \prod_{\lambda \text{ eigenvalues} \text{ of } M} (1 - \lambda + z\lambda).$$

Corollary 2.5. For symmetric kernels $M$ of a determinantal processes, the variance of the number of particles is equal to the sum variances of independent Bernoulli($\lambda_j$) random variables with bias, where $0 \leq \lambda_j \leq 1$ are the nonzero eigenvalues of $M$.

Proof.

$$\Var(\text{number of particles}) = Tr(M(I - M)) = \sum_{\lambda \text{ eigenvalues of } M} \lambda(1 - \lambda)$$

$$\sum_{\lambda \text{ eigenvalues of } M} \lambda(1 - \lambda) = \sum_{\lambda \text{ eigenvalues of } M} \Var(\text{Bernoulli(}\lambda\text{))}.$$ 

Fact: If $\{x_i\}_{i=1,\ldots,\infty}$ are independent Bernoulli variables with possible different biases $p_i$ but $\sum_{i=1}^{\infty} p_i(1 - p_i) = \infty$, then partial sums tend to be gaussian random variables.

3 Height fluctuations

In this section, we show that the measure on the set of all dimer coverings of infinite honeycomb converges to a probability measure $\mu_{a,b,c}$, which is a determinantal point process on $\mathbb{Z}$ with kernel $K^{-1}$. Then we calculate the variance of height function.
The matrix $K^{-1}(w_i, b_j)$ only depends on $i - j$. More precisely,

$$K^{-1}(w_i, b_j) = \begin{cases} \frac{-\sin((i-j)\theta_a)}{\pi(i-j)a} & i - j \neq 0 \\ \frac{\varphi_a}{\pi a} & \text{otherwise}. \end{cases}$$

Given any set of $k$ edges of type $a$ in the vertical column passing through the edge $w_{0,0}$, each of these $k$ edges either appears in the column or not. The presence of the edges in the column forms a determinantal process with the kernel $K^{-1}(w_i, b_j)_{1 \leq i,j \leq k}$ which is the discrete sine kernel.

**Remark 3.1.** If $M \in U(n)$, the eigenvalues of $M$ lies on a unit circle. So the eigenvalues of a random $M$ in $U$ form a determinantal process with kernel $K_n$. As $n \to \infty$, $K_n \to K$, we have

$$K_{ij} = \frac{\sin((y_i - y_j)a)}{\pi(y_i - y_j)a}$$

after rescaling where $y_i, y_j$ are angles.

**Definition 3.2.** Given a dimer cover $M$ on a honeycomb graph, the height function assigned to the vertices of the lozenge tiling (or the face of the honeycomb) is defined by the following rule: Start with 0 at some arbitrary vertex. Along any edge of the Lozenge tiling which has a black vertex on the left, the height increases by 1. Along any edge with a white vertex to its left, the height decreases by 1.
The height change \( h(v_1) - h(v_2) \) from \( v_1 \) to \( v_2 \) is

\[
h(v_1) - h(v_2) = -L + 3d
\]

where \( L \) is the number of horizontal edges and \( d \) is the number of horizontal dimers between \( v_1 \) and \( v_2 \). If \( a, b, c \) satisfying the triangle inequality and \( \theta_a \) is the opposite angle of \( a \). Note that the expected value of \( d \) is

\[
E(d) = \frac{\theta_a}{\pi} |v_1 - v_2| = \frac{\theta_a L}{\pi}.
\]

Then the height fluctuation \( \text{Var}(h) \) between two vertices \( v_1, v_2 \) is equal to \( \text{Var}(d) \) because the height function is linear of \( d \). As discussed above, the presence of the edges in the column forms a determinantal point process with kernel \( M_L = K^{-1}(w_i, b_j)_{1 \leq i, j \leq L} \). Thus, by 2.5,

\[
\text{Var}(h) = \text{Var}(d) = \sum_{\lambda \text{ eigenvalues of } M_L} \lambda (1 - \lambda) = \text{Tr}(M_L(I - M_L))
\]

Let \( a_{i-j} = M_{i,j} \) for \( 1 \leq i, j \leq k \). By the Fourier transform of the function \( f(\theta) = |\theta| (\pi - |\theta|) \) for \( \theta \in [-\pi, \pi] \), we have

\[
\text{Tr}(M_L(I - M_L)) = L\alpha_0(1 - \alpha_0) - a_1^2(2L - 2) - a_2^2(2L - 4) - \cdots - a_{L-1}^2\alpha_{L-1}
\]

\[
= \frac{1}{\pi^2} \log L + O(1).
\]

Therefore, we can conclude that the height difference between \( v_1, v_2 \) tends a Gaussian when \( v_1, v_2 \) are far apart. One can prove that the scaling limit of height function is the Gaussian free field.