Math 113: Course Summary

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September 25, 2009

General Information: Math 113 is a course in real analysis. It is the first half of the undergraduate series in real analysis, M113-4. There is some overlap between M113 and M101. It is not necessary to take M101 before taking M113, though some students feel that M101 is a good preparation for the more difficult M113. If you don't have any idea what real analysis is, you can read my summary of M101.

M113 essentially covers 3 topics.

- Point set topology in metric spaces, especially as applied to $\boldsymbol{R}^n$.
- Measure theory and the Lebesgue integral.
- The beginnings of functional analysis – e.g. Fourier series.

Metric Spaces: In M101 you pretty much stick to the real line when discussing concepts such as continuity, connectedness, and compactness. In M113, you consider these ideas in a more general setting. A *metric space* is a set X together with a *distance function* $d : X \times X \to \boldsymbol{R}$ which satisfies the following properties:

- $d(x, y) > 0$ for all $x \neq y$ and $d(x, x) = 0$ for all x.
- $d(x, y) = d(y, x)$.
- $d(x, y) \leq d(x, z) + d(y, z)$.

For instance, $\boldsymbol{R}$ is a metric space with the metric $d(x, y) = |x - y|$. More generally, $\boldsymbol{R}^n$ is a metric space with the metric

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}.$$

An Important Example: Lets $L_2(\boldsymbol{N})$ denote the space of infinite sequences $\{a_i\}$ such that

$$\sum_{i=1}^{\infty} a_i^2 < \infty,$$

and one defines

$$d(\{a_i\}, \{b_i\}) = \sqrt{\sum_{i=1}^{\infty} (a_i - b_i)^2}.$$

The space $L_2(\boldsymbol{N})$ is an infinite-dimensional version of $\boldsymbol{R}^n$.

Limits and Convergence: The notion of convergence in a metric space works just about the same way as it does in $\boldsymbol{R}$. A sequence $\{x_n\}$ in a metric space X is said to *converge* to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$. A sequence $\{x_n\}$ is said to be *Cauchy* if, for all each integer $N > 0$, there is some integer M such that $m, n > M$ implies that $d(x_m, x_n) < 1/N$.

The space X is said to be *complete* if every Cauchy sequence in X converges to a point in X. In our summary for M101 we defined $\boldsymbol{R}$ as the set of equivalence classes of Cauchy sequences in $\boldsymbol{Q}$. One might say that $\boldsymbol{R}$ is the *completion* of $\boldsymbol{Q}$. In general, one can start with any metric space X and form a complete metric space X^* of equivalence classes of Cauchy sequences in X. The space X^* is called the *completion* of X.

A nice example of a complete metric space is $L^\infty(\boldsymbol{R})$, the space of bounded continuous functions on $\boldsymbol{R}$. The metric is

$$d(f, g) = \sup |f - g|.$$

Considering this space in detail, one can prove Picard's theorem, the existence and uniqueness result for first order differential equations. This is one of the applications you will see in M113.

Continuity: Given metric spaces X and Y, a map $f : X \to Y$ is said to be *continuous at* x if it satisfies the following property: If $\{x_n\}$ is any sequence in X that converges to $x \in X$ then $\{f(x_n)\}$ is a sequence in Y that converges to $f(x)$. The map f is said to be *continuous* if it is continuous at all $x \in X$. One can compose maps between metric spaces just like one composes real valued maps. Given $f : X \to Y$ and $g : Y \to Z$, we have the composition $h : X \to Z$, where $h(x) = g(f(x))$. The map h is continuous provided that both f and g are continuous. What is nice about this general result is that you just have to prove it once, for a metric space, and then it applies to all the examples of metric spaces.

Point Set Topology: Let X be a metric space and let $x \in X$ be a point. We define the ball

$$B_r(x) = \{y \in X |\ d(x,y) < r\}.$$

This is the generalization of an open interval on $\boldsymbol{R}$. A subset $U \subset X$ is called *open* if it has the following property. For every $x \in U$ there is some $r = r_x$ such that $B_r(x) \subset U$ as well. The set $C \subset X$ is called *closed* if the complement $X - C$ is open.

A subset $K \subset X$ is called *compact* if it has the following property: Suppose $\mathcal{U}$ is a collection of open sets whose union contains K. Then there is some finite collection $U_1, ..., U_n \in \mathcal{U}$ whose union contains K. (This is supposed to hold for any such covering $\mathcal{U}$.) Compactness is a very important concept in analysis. In M113, you see a number of results about compact sets:

- A subset of $\boldsymbol{R}^n$ is compact if and only if it is closed and bounded.
- Any infinite sequence of points in a compact set K has a convergent subsequence. The limit of this subsequence lies in K.
- Every continuous function $f : K \to \boldsymbol{R}$ attains its extreme points on K. In particular, a positive continuous function on a compact set has a positive minimum.
- If $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K)$ is compact in Y.

Flaws in the Riemann Integral: At some point in calculus, you learned about Riemann integral. This magical construction works great for continuous functions. However, for functions that aren't continuous, the Riemann integral has some serious flaws.

For example, consider the function $f : [0,1] \to \boldsymbol{R}$ such that $f(x) = 1$ if x is irrational and $f(x) = 0$ if x is rational. The Riemann integral does not assign a value to

$$\int_0^1 f\ dx$$

because the upper and lower sums involved do not converge to the same thing. However, our function f agrees with the constant function $g \equiv 1$ except on a countable set of points, so one would expect "the integral" to be 1.

More generally, the Riemann integral has trouble with functions that have discontinuities at an infinite collection of points. Measure theory and the associated Lebesgue integral are able to deal with a very broad class of such functions. Moreover, on every function that has a Riemann integral, the Lebesgue integral gives the same answer. So, the Lebesgue integral is a kind of enhancement of the Riemann integral.

Measure Theory: The framework for measure theory is quite general. One has a triple $(X, \mathcal{A}, \mu)$, where X is a set, and $\mathcal{A}$ is a distinguished family of subsets called a *σ-algebra*, and $\mu : \mathcal{A} \to [0, \infty)$ is a map. $\mathcal{A}$ satisfies the following axioms:

- $\mathcal{A}$ contains the empty set $\emptyset$.
- If S belongs to $\mathcal{A}$ then so does $X - S$.
- If $\{S_i\}$ is a countable collection of sets in $\mathcal{A}$ then the union $\bigcup S_i$ also belongs to $\mathcal{A}$.

The measure μ satisfies the following axioms.

- $\mu(\emptyset) = 0$.
- $\mu(S) \geq 0$ for all S in $\mathcal{A}$.
- If $\{S_i\}$ is a disjoint collection of sets in $\mathcal{A}$ then

$$\mu\Big(\bigcup S_i\Big) = \sum \mu(S_i).$$

An important special case is the triple $(\boldsymbol{R}^n, \mathcal{B}, \mu)$. Here $\mathcal{B}$ is the smallest σ-algebra that contains all the open sets and μ is the extension of Euclidean volume. One obtains a typical set in $\mathcal{B}$ by taking countable unions and intersections of open sets, a fininte number of times. For instance, the unit ball minus all rational points is a Borel set.

It turns out that the notion of volume extends to a measure on $\mathcal{B}$. The σ-algebra $\mathcal{B}$ is called the *Borel σ-algebra*. The fact that volume extends to a measure on $\mathcal{B}$ suggests the existence of a more powerful integral. After all, one of the main functions of an integral is to compute volumes, and here we are saying that one can assign a volume to any Borel set.

A closely related example is the *Lebesgue σ-algebra*. A set $S \subset \boldsymbol{R}^n$ is said to have *measure zero* if, for every $\epsilon > 0$, we have $S \subset T_\epsilon$, where T_ϵ is a Borel set such that $\mu(T_\epsilon) < \epsilon$. A Lebesgue set is one of the form $B \cup C$, where B is a Borel set and C has measure zero.

The Lebesgue Integral Let $(X, \mathcal{A}, \mu)$ be a triple as above. Here we will describe the Lebesgue integral for this triple. One builds this integral from the ground up, in a completely canonical way.

Let S be a set in $\mathcal{A}$. The *characteristic function* for S is the function $I_S : X \to [0,1]$ such that $I_S(x) = 1$ if $x \in A$ and otherwise $I_S(x) = 0$. One defines

$$\int_X I_S = \mu(S).$$

A *simple function* is a finite positive sum

$$f = \sum_{i=1}^{n} a_i I_{S_i},$$

where $S_1, ..., S_n$ are sets in $\mathcal{A}$ and $a_1, ..., a_n$ are positive reals. One defines

$$\int_X f = \sum_{i=1}^{n} a_i \mu(S_i).$$

Let $f : X \to \boldsymbol{R}$ be a non-negative function. We define

$$\int_X f = \sup_{g \le f} \int X_g.$$

The supremum is taken over all simple functions g such that $g(x) \le f(x)$ for all $x \in X$.

If f is a function that takes on both positive and negative values, we may write $f = f_+ - f_-$, where f_+ and f_- are both non-negative functions. We defune

$$\int_X f = \int_X f_+ - \int_X f_-.$$

This defines the Lebesgue integral for any real-valued function on X.

A function f is called *measurable* if $f^{-1}(I)$ is in $\mathcal{A}$ for any interval I. The Lebesgue integral is well behaved on all measurable functions. One of the things you do in M113 is prove a number of theorems about the behavior of the Lebesgue integral on measurable functions. For instance, if $\{f_i\}$ is any nondecreasing sequence of bounded measurable functions and $f = \sup f_j$, then

$$\int_X f = \sup\Big(\int_X f_j\Big).$$

This is known as the *Monotone Convergence Theorem.*

As a special case, you can do all this for the Lebesgue sets in $\boldsymbol{R}^n$. Any reasonable construction (i.e., one that does not use the axiom of choice in an essential way) produces a measurable function. Thus, the Lebesgue integral makes sense and behaves well for practically any function you can construct. Moreover, the Lebesgue and Riemann integrals agree whenever the Riemann integral is well-defined.

Hilbert Space: An *inner product* on a vector space V is a binary operation $\langle\ ,\ \rangle$ such that

- $\langle v, v\rangle \geq 0$ with equality if and only if $v = 0$.
- $\langle v, w\rangle = \langle w, v\rangle$.
- $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$.

The inner product defines a metric on V as follows:

$$d(v, w) = \langle v - w, v - w\rangle.$$

V is called a *Hilbert space* if V is complete with this metric. The space $L_2(\boldsymbol{N})$ above is the prototypical example of a Hilbert space.

Equipped with the Lebesgue integral, we can generalize this example in a useful way. Let V denote the vector space of Lebesgue measurable functions $f : \boldsymbol{R} \to \boldsymbol{R}$ such that

$$\int_R f^2 < \infty.$$

We write $f \sim g$ is f and g agree except on a set of measure 0. In this case, these functions have the same integral. We let $L_2(\boldsymbol{R}, \boldsymbol{R}) = V/\sim$, the quotient vector space. We define

$$\langle f, g\rangle = \int_{\boldsymbol{R}} fg.$$

This is well defined on $L_2(\boldsymbol{R}, \boldsymbol{R})$ and makes $L_2(\boldsymbol{R}, \boldsymbol{R})$ into a Hilbert space.

The construction we gave has some variants. For instance, one can extend the Lebesgue integral to complex valued functions. One just integrates the real and imaginary parts separately. We can then define $L_2(\boldsymbol{R}, \boldsymbol{C})$ as above, using complex-valued functions in place of real functions. The inner product is given by

$$\langle f, g \rangle = \int_{\boldsymbol{R}} f\overline{g}.$$

That is, we take the complex conjugate of g.

Going in another direction, we can replace $\boldsymbol{R}$ with $\boldsymbol{R}/\boldsymbol{Z}$, the unit circle. That is, we can look at 1-periodic functions and define the inner product by integrating over a single period. This leads to the space $L_2(\boldsymbol{R}/\boldsymbol{Z}, \boldsymbol{C})$. Similarly, in the example we gave in the very beginning of this summary, we can replace $\boldsymbol{N}$ with $\boldsymbol{Z}$ and consider $L_2(\boldsymbol{Z}, \boldsymbol{C})$, the Hilbert space of bi-infinite complex-valued sequences.

Fourier Series: Let $L_2 = L_2(\boldsymbol{R}/\boldsymbol{Z}, \boldsymbol{C})$ and $l_2 = L_2(\boldsymbol{Z}, \boldsymbol{C})$. Let g_n be the function

$$g_n(x) = \exp(2\pi i n x).$$

It turns out that every $f \in L_2$ gives rise to a unique sequence $\{c_i\} \in l_2$ such that

$$f \sim \sum_{i=-\infty}^{\infty} c_n g_n.$$

In other words, the two functions agree up to a set of measure zero. Thus, the sum on the right defines the same element in L_2. This sum is called the *Fourier series* of F.

One of the most beautiful facts about the Fourier series is that

$$\int_{\boldsymbol{R}} |f|^2 = \sum_{n=-\infty}^{\infty} |c_i|^2.$$

That is, the map $f \to \{c_n\}$ is an *isometry* (distance preserving map) from L_2 to l_2.