Math 114: Course Summary

Rich Schwartz

September 25, 2009

General Information: Math 114 is a course in real analysis. It is the second half of the undergraduate series in real analysis, M113-4. In M113, you examine concepts such as limits, continuity, and integration in a much more general setting. In M114, you examine concepts such as partial derivatives, line integrals (and their generalizations) and Stokes' theorem in a more general setting. The main topics in M114 are

- Partial derivatives, especially the Inverse Function Theorem and the Implicit Function Theorem.
- Basic information about manifolds and calculus on manifolds, with special emphasis on differential forms.
- The general version of Stokes' theorem.

I'll discuss these items in turn.

Partial Derivatives: In M114, you take a much more natural and general approach to partial derivatives than is taken in a typical calculus course like M18. Let $F : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be a map whose partial derivatives exist and vary continuously. For each point p in the domain of F, there is a linear map $dF_p : \boldsymbol{R}^n \to \boldsymbol{R}^m$ such that

$$\|F(p + \epsilon v) - F(p) + \epsilon \; dF_p(v)\| < O(\epsilon^2),$$

for ϵ sufficiently small. That is, in a neighborhood of p, the map F looks like the linear map dF to second order. The map dF is just the matrix of first partial derivatives of F. In M114 you will give a rigorous proof of this result.

If $F : \boldsymbol{R}^n \to \boldsymbol{R}^m$ and $G : \boldsymbol{R}^m \to \boldsymbol{R}^p$ are two maps, we have the composition $H = G \circ F$, defined as $H(p) = G(F(p))$. In M114 you prove the general form of the Chain Rule:

$$dH_p = dG_{F(p)}.dF_p.$$

Here the "dot" means the matrix product. This equation is supposed to hold for all $p \in \boldsymbol{R}^n$. For convenience, I have stated things for maps that are defined on all Euclidean space, but the same result holds for maps that are just defined on open subsets of Euclidean space.

The Inverse Function Theorem: Let U and V be two open sets in $\boldsymbol{R}^n$. A map $F : U \to V$ is called a *diffeomorphism* if

- F is a bijection.
- F is smooth. All partial derivatives of all orders exist.
- F^{-1} is smooth.
- dF_p is a linear isomorphism for all $p \in U$.

The Inverse Function Theorem gives a criterion for F to be a (local) diffeomorphism.

Theorem 0.1 *Suppose $F : U \to \boldsymbol{R}^n$ is defined and smooth in a neighborhood U of 0. Suppose also that dF_0 is nonsingular. Then there is a neighborhood $U' \subset U$ such that $F : U' \to V'$ is a diffeomorphism. Here $0 \in U'$ and $V' = F(U')$.*

So, to check that F is a diffeomorphism in a neighborhood of 0 one just needs to compute the matrix of partials at 0. Of course, a similar result can be formulated with 0 replaced by any point $p \in \boldsymbol{R}^n$. In M114 you see the proof of the Inverse Function Theorem. The Inverse Function Theorem is of crucial importance when it comes time to study manifolds.

The Implicit Function Theorem: The Implicit Function Theorem is a variant of the Inverse Function Theorem. Let $m < n$ and let $k = n - m$. We write $\boldsymbol{R}^n = \boldsymbol{R}^k \times \boldsymbol{R}^m$. Given a smooth map $g : \boldsymbol{R}^k \to \boldsymbol{R}^m$, we can form the *graph*

$$\Gamma_g = \{(x, g(x)) |\ x \in \boldsymbol{R}^k\}.$$

More generally, if g is just defined in an open subset $U \subset \boldsymbol{R}^k$, then Γ_u is contained in the subset $U \times \boldsymbol{R}^m \subset \boldsymbol{R}^n$.

Theorem 0.2 *Suppose U is a neighborhood of 0 in $\boldsymbol{R}^n$ and $F : U \to \boldsymbol{R}^m$ is a smooth map. Suppose that $F(0) = 0$ and dF_0 is a linear isomorphism when restricted to $\{0\} \times \boldsymbol{R}^m$. Then there is a smooth map $g : \boldsymbol{R}^k \to \boldsymbol{R}^m$ and a neighborhood U' of 0 in $\boldsymbol{R}^n$ such that*

$$F^{-1}(0) \cap U' = \Gamma_g \cap U'.$$

That is, F vanishes exactly on the graph of g.

Special cases of the Implicit Function Theorem are familiar from calculus. For instance, suppose that $F : \boldsymbol{R}^3 \to \boldsymbol{R}$ is a map such that the gradient $\nabla F(0)$ is nonzero and $F(0) = 0$. Then, in a neighborhood of 0, the set $F^{-1}(0)$ is a surface. To see how this follows from the Implicit Function Theorem, we take $k = 2$ and $m = 1$ and $n = 3$. We rotate so that $dF/dz \neq 0$. Then dF restricted to $\{0\} \times \boldsymbol{R}^1$ is just dF/dz. This is nonzingular. But, in this case, Γ_g is just the ordinary graph of g. So, in a neighborhood of $0 \in \boldsymbol{R}^3$, the map F vanishes on the graph of a smooth function – i.e.a surface.

We stated the Implicit Function Theorem in a particular way, for the sake of convenience. A more natural way is that F vanishes along a k-dimensional surface in a neighborhood of 0 provided that $F(0) = 0$ and dF_0 is onto. When dF_0 is onto, we can rotate $\boldsymbol{R}^n$ in such a way that dF_0 is an isomorphism when restricted to $\{0\} \times \boldsymbol{R}^m$.

Embedded Submanifolds: Let $S \subset \boldsymbol{R}^n$ be a set. We say that S is a *k-dimensional submanifold* if, for every point $p \in S$, there is a neighborhood U of p and a smooth map $F : U \to \boldsymbol{R}^m$ such dF_p is onto. Here $k + m = n$, as above. By the Implicit Function Theorem, S is a suitably rotated graph Γ_g of a smooth map $g : \boldsymbol{R}^k \to \boldsymbol{R}^m$. The map F depends on the point p, but the same map may be used for all points of S sufficiently close to p. In other words S looks locally like a (slightly distorted) copy of $\boldsymbol{R}^k$.

The concept of an embedded submanifold is also familiar from calculus. As a special case, suppose we have a smooth map $F : \boldsymbol{R}^3 \to \boldsymbol{R}$ such that ∇F_p is nonzero for all p such that $F(p) = 0$. Then the level set $F^{-1}(0)$ is a smooth surface. Curves in the plane furnish another example, though in this case it is easier just to say that a curve is the image of a smooth map $F : \boldsymbol{R} \to \boldsymbol{R}^2$ such that F' is never 0.

In M18 you learn about line integrals over curves and also about surface integrals. Though it is not often mentioned in a calculus class, the *integrand* of a line integral – meaning the thing you integrate – is a 1-form. The integrand of a surface integral is a 2-form. These concepts extend naturally to the concept of a k-form, which you can integrate over a k-dimensional submanifold. In M114, you develop the theory of k-forms. Part of this theory is algebraic and part is analysis-based. I'll explain the algebraic part first and then the analysis-based part.

Exterior Algebra: A k-*tensor* on $\boldsymbol{R}^n$ is a map

$$T : \boldsymbol{R}^n \times ... \times \boldsymbol{R}^n \to \boldsymbol{R}$$

which is linear in each of the k positions. For instance

$$T(v_1 + rv_1', v_2, ..., v_k) = T(v_1, ..., v_k) + rT(v_1', v_2, ..., v_k),$$

and similiarly for the other positions. T is called *alternating* if

$$T(..., v_i, ..., v_j, ...) = -T(..., v_j, ..., v_i, ...).$$

That is, switching two of the vectors puts a minus sign in front of the value of T. The most familiar object like this is the determinant, which one can think of as an alternating n-tensor on $\boldsymbol{R}^n$.

The space of all alternating k-tensors on $\boldsymbol{R}^n$ is denoted by $\Lambda^k \boldsymbol{R}^n$. This space is known as the k-*th exterior power* of $\boldsymbol{R}^n$. The space $\Lambda^k \boldsymbol{R}^n$ is naturally a vector space of dimension n choose k. Here is the beauty of elements in $\Lambda^k \boldsymbol{R}^n$. Suppose that $\Pi \subset \boldsymbol{R}^n$ is an oriented k-plane and $\omega \in \Lambda^k \boldsymbol{R}^n$. Then we can choose a basis for Π, compatible with the orientation, that spans a parallelopiped of unit volume. Let $\{v_j\}$ be this basis. We define

$$\omega(\Pi) = \omega(v_1, ..., v_k).$$

At first it looks like this depends on the basis. However, if we pick a different basis $\{v_j'\}$ with all the same properties then it turns out that

$$\omega(v_1, ..., v_k) = \omega(v_1', ..., v_k').$$

In other words, ω assigns a well defined number to each oriented k-plane.

The Wedge Product: It turns out that there is a natural map, denoted by $\wedge$, from $\Lambda^k \boldsymbol{R}^n \times \Lambda^m \boldsymbol{R}^n$ to $\Lambda^{k+m} \boldsymbol{R}^n$. In other words, we can take an alternating k form and an alternating m form and produce an alternating $k + m$

form. The operation is traditionally known as the *wedge product*. Here is the formula for this operation.

$$\omega_1 \wedge \omega_2(v_1, ..., v_k, w_1, ..., w_m) =$$

$$\frac{1}{(k+m)!} \sum_\sigma \text{sign}(\sigma)\ \omega_1(\sigma(v_1), ..., \sigma(v_k)) \times \omega_2(\sigma(w_1), ..., \sigma(w_m)).$$

Here $\times$ is just multiplication. The sum takes place over all permutations of $(k+m)$ elements. The sign of the permutation σ is 1 if σ is an even permutation and -1 if σ is an odd permutation.

This formula can be extended to k-tuples of forms giving (in the case $k = 3$) expressions like $\omega_1 \wedge \omega_2 \wedge \omega_3$. In particular, if $\omega_1, ..., \omega_k$ are all 1-tensors (i..e. linear functionals) on $\boldsymbol{R}^n$, then $\omega_1 \wedge ... \wedge \omega_k \in \Lambda^k \boldsymbol{R}^n$. These particular elements form a basis for $\Lambda^k \boldsymbol{R}^n$. Any $\omega \in \Lambda^k \boldsymbol{R}^n$ has a unique decomposition

$$\omega = \sum_I c_I \omega_I; \qquad I = (i_1, ..., i_k); \qquad \omega_I = dx_{i_1} \wedge ... \wedge dx_{i_k}.$$

Here dx_i is the linear functional that is 1 on the standard basis vector e_i and 0 on all other basis vectors e_j. The sum takes place over all multi-indices with $1 \leq i_1 < i_2 < ..., < i_k \leq n$.

Differential Forms: A *differential k-form* on $\boldsymbol{R}^n$ is a smooth choice of an element $\omega(p) \in \Lambda^k \boldsymbol{R}^n$ for each $p \in \boldsymbol{R}^n$. By *smooth* we mean that, if we write $\omega(p)$ in terms of a basis of $\Lambda^k \boldsymbol{R}^n$ at each point, then the coefficients are smooth functions on $\boldsymbol{R}^n$ in the ordinary sense. You have probably seen a related concept: A vector field on $\boldsymbol{R}^n$ is smooth if the component functions are smooth. The idea here is similar.

Here we will give an informal idea how to integrate a smooth k-form ω over a smooth k-dimensional submanifold Ω. The way we explain it is not really how it is done in M114, but it does give some intuition for what is going on.

First we subdivide Ω into a large number of small k-dimensional tetrahedra. These tetrahedra are drawn on Ω, so to speak, and consequently not quite flat. However, we replace each tetrahedron by a flat tetrahedron that has the same vertices. We now have a new "faceted surface" consisting of a bunch of small tetrahedra. This faceted surface is a close approximation to Ω. Suppose that $T_1, ..., T_m$ is our list of tetrahedra. Let $\Pi_1, ..., \Pi_m$ be the k-planes that contain these tetrahedra. We define

$$I = \sum_{j=1}^{m} \omega(\Pi_j)\text{volume}(T_j).$$

The value of I depends on the choice of tetrahedra, much in the same way that a Riemann sum depends on the choice of partition. However, as we let the mesh size of the tetrahedra tend to 0, we get a well defined limit

$$\int_\Omega \omega.$$

Again, the procedure we just described is not the actual one taken in M114. In M114 the integral is defined carefully, in terms that have more to do with the change of variables formula from multivariable calculus. Ω.

The d Operator: Let $X_{k,n}$ denote the space of differential k-forms on $\boldsymbol{R}^n$. There is a natural map $d : X_{k,n} \to X_{k+1,n}$. We define

$$d(f dx_{i_1} \wedge ... \wedge dx_{i_k}) = \sum_{j=1}^{n} \frac{df}{dx_j} dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}.$$

Any differential k-form is a sum of these, and so we can extend d linearly to all of $X_{k,n}$.

Some special cases are well-known in physics.

- The map $X_{0,3} \to X_{1,3}$ can be interpreted as the gradient.
- The map $X_{1,3} \to X_{2,3}$ can be interpreted as the curl.
- The map $X_{2,3} \to X_{3,3}$ can be interpreted as the divergence.

In order to make these interpretations, one has to identify $X_{1,3}$ and $X_{2,3}$ with the space of smooth vector fields on $\boldsymbol{R}^3$. This is possible to do.

d satisfies the famous equation

$$d \circ d = 0.$$

In the classical cases above, this equation corresponds to various physical statements such as "the divergence of the curl is zero."

Stokes' Theorem: Now for the highlight of M114. Stokes' theorem deals with the situation where Ω is a k-dimensional submanifold in $\boldsymbol{R}^n$ and the boundary $\partial\Omega$ is a $(k-1)$ dimensional submanifold. You should picture something like a potato chip in space for the case $n = 3$ and $k = 2$. If ω is a differential $(k-1)$ form then $d\omega$ is a differential k form. It makes sense to integrate ω over $\partial\Omega$ and it also makes sense to integrate $d\omega$ over Ω. Stokes' Theorem says

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega.$$

All the classical versions of Stokes' theorem fit into this one general statement. When $n = 2$ and $k = 2$ this is Green's Theorem. When $n = 3$ and $k = 2$ this is the classical Stokes' theorem. When $n = 3$ and $k = 3$ this is the divergence theorem.

A Final Word: It might look like Stokes' theorem is the culminating result in a long of line of mathematics, but it is actually just the starting point for the theory of *de Rham cohomology* on smooth manifolds. You can learn about this in a graduate course in differential topology.