Here are some problems:

1.: A *Hopf circle* is defined to be a set of the form

$$\{(\cos(\theta) + i\sin(\theta))q\}, \qquad \theta \in [0, 2\pi].$$

Here q is a unit quaternion. Prove that every point of S^3 lies on a Hopf circle and that every two Hopf circles are disjoint and linked.

2: For each of the quaternions $\ell = i, j, k$ describe the rotations of S^2 defined by the quaternion $q = \cos(\theta) + \ell\sin(\theta)$ as θ ranges from 0 to 2π. In other words, you want to see that the transformation T_q does on the set of pure quaternions, where $T_q(p) = qpq^{-1}$.

3: Recall from class that the map $q \to T_q$ gives a map from S^3 to $SO(3)$, the group of rotations of S^2. Use your answer from part 2 (and other arguments) to prove that this map is onto. In other words, every rotation of S^2 is represented by a unit quaternion (in two ways).

4: Prove that the map from $S^3 \times S^3$ to $SO(4)$ given by $(q_1, q_2) \to T_{q_1,q_2}$ is onto. Here

$$T_{q_1,q_2}(a) = q_1 a q_2^{-1}.$$

In this formula, a is any quaternion.

5: Concretely, the 24 cell is the convex hull of the vectors

$$(\pm 1, \pm 1, 0, 0)$$

and their permutations. Prove that this polytope has 24 faces, is self-dual, and tiles $\boldsymbol{R}^4$. To say that the polytope is self-dual is to say that if you place a vertex at the center of each face, you get the same polytope back up to rotations and dilations.

6: Prove that any isometry of $\boldsymbol{R}^n$ is the product of a finite number of reflections in hyperplanes.

7: Take as many of Frank Farris's planar patterns as you can, and work out the symmetry group and quotient orbifold.

8: Draw a self-portrait on at least 5 different planar Euclidean orbifolds.

9: There are 17 topologically distinct planar orbifolds. Prove that there are more than 50 topologically distinct 3-dimensional spatial orbifolds. There are actually 230 of them.

10: Consider the surface made by gluing parallel sides of a block letter L together in a fairly obvious way. (A block L is a big square with a small square taken out of its top left corner.) What is the genus of this surface and how many cone points does it have? Find an affine automorphism of this surface. (A few people will be experts at this...)

11: Suppose that you have a finite set of tiles with the following property. For each $R > 0$ there is a way to tile a region which covers the disk of radius R centered at the origin using only isometric copies of tiles from this set. Prove that you can tile the whole plane with isometric copies of the tiles.

12: Flesh out the argument in class to show that for any binary sequence there is some Penrose tiling T_0 and some point p which has this sequence as its containment sequence. Recall that the containment sequence $b_0, b_1, ...$ works like this. There is an infinite family $T_1, T_2, ...$ of Penrose tilings such that T_{k+1} is the inflation of T_k. The number b_k is 0 if p lies in a kite and 1 if p lies in a kite. Exercise 11 might be helpful here.

13: Use problem 13 to show that there are uncountably many distinct equivalence classes of Penrose tilings, where two tilings are declared equivalent if they are translation equivalent.

14: As in class, let $\mathcal{T}$ be the pointed Penrose tiling space. Members of $\mathcal{T}$ are pairs (T, x) where T is a Penrose tiling, rotated so that some of the tiles have edges parallel to the x-axis, and x is a point in the plane. Define the distance between (T_1, x_1) and (T_2, x_2) as follows. Translate these tilings so that x_1 and x_2 are both the origin. The the distance is 2^{-k} where k is the largest number of layers about the origin on which these translated tilings agree. Discuss the local structure of $\mathcal{T}$ with this metric.