Complex Numbers

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1 From Natural Numbers to Reals

You can think of each successive number system as arising so as to fill some deficits associated with the previous one. Let's consider some number systems in turn.

The Natural Numbers: The set of natural numbers, $\{1, 2, 3, 4, ...\}$, is denoted by $\boldsymbol{N}$. You can order, add, and multiply numbers in $\boldsymbol{N}$ and there are various compatibilities between these operations. For instance $a(b + c) = ab + ac$. However, subtraction is tricky. You can only take $a - b$ when $a > b$.

The Integers The set of integers $\{..., -2, -1, , 0, 1, 2, ...\}$ is denoted by $\boldsymbol{Z}$. You can order, add, subtract, and multiply numbers in $\boldsymbol{Z}$ with the same compabitilities. However, division is tricky. You can only take a/b when b evenly divides a.

The Rationals The rationals are denoted $\boldsymbol{Q}$. They are the expressions of the form p/q where p and q are integers, with $q \neq 0$. Two such expressions p_1/q_1 and p_2/q_2 are meant to be the same (or, techhically, *equivalent*) if $p_1q_2 = p_2q_1$. For instance 2/3 and 4/6 are equivalent. The set of rationals forms a *ordered field*, which means there is an ordering on the rationals, and you can do all 4 basic arithmetic operations, and you have the usual compatibilities between all the operations. However, in the rationals you cannot generally take limits, perform square roots, take logs, etc. All this stems from the fact that the rationals have "gaps" in them.

The Reals: The reals are denoted $\boldsymbol{R}$. As in class, you can define a real number as a Dedekind cut (A, B), where A and B are both sets of rationals satisfying certain axioms. The reals remain an ordered field, but they have the additional virtue that the real number system has no gaps. This is sometimes formalized by saying that the real numbers have the *least upper bound* property. If $S \subset \boldsymbol{R}$ is any set which is bounded from above, then there is some b such that $a \leq b$ for all $a \in S$, and no smaller b has this property. Once you have the least upper bound property, you can define square roots, limits, logs, etc.

2 Roots of Polynomials

The real numbers also have a deficit of sorts. You cannot take the square root of a negative number. Put another way, some polynomials do not have real roots. The classic example is $x^2 + 1$. The same problem persists with other polynomials. For instance, the quadratic equation

$$Ax^2 + Bx + C$$

has solutions which come from the quadratic formula:

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

But, there are problems if $B^2 - 4AB < 0$. In this case, the formula doesn't give real numbers.

3 The Complex Numbers

The complex numbers are defined so that they overcome the deficits mentioned for the real numbers. The main idea is to introduce the symbol i, which satisfies the rule that

$$i^2 = -1.$$

It is important to remember that i is just a made-up symbol. It doesn't have any properties at all besides the ones which we give it.

A complex number is an expression of the form

$$a + bi,$$

where a and b are real numbers. The set of complex numbers is denoted $\boldsymbol{C}$.

Here are the basic laws for complex numbers:

- Addition: $(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$.
- Subtraction: $(a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$.
- Multiplication:

  $$(a_1 + ib_1) * (a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1).$$

  Basically, you just multiply the expression out as you would in high school algebra, and then simplify the expression using the rule that $i^2 = -1$.

- Inversion: $1/(a + bi)$ is defined to be

  $$\frac{a}{a^2 + b^2} - \left(\frac{b}{a^2 + b^2}\right)i.$$

  Note that $1/(a + bi)$ times $a + bi$ equals 1.

- Division

  $$\frac{a + bi}{c + di} = (a + bi) \times \left(\frac{1}{c + di}\right).$$

  This is using inversion and multiplication to define division.

Once all these definitions are made, the complex numbers also form a field. That is, the usual compatibility rules for the operations hold.

There are several other important operations on complex numbers:

- Norm: $|a + ib|$ is defined to be $\sqrt{a^2 + b^2}$. Geometrically, $|a + bi|$ is the distance from the point (a, b) to the origin in the plane.
- Conjugation: $\overline{a + bi}$ is defined to be $a - bi$. The numbers $a + bi$ and $a - bi$ are called *conjugates*.

These last two operations make some of the formulas simpler. It is customary to write complex numbers as single variables, such as $z = a + bi$. If s is a real number, then $z/s = (a/s) + (b/s)i$. In general, if w is another complex number

$$z/w = \frac{z\overline{w}}{|w|^2}.$$

This gives a more concise formula for multiplication.

4 Goodbye Order

The only thing that we really lose when we use complex numbers in place of real numbers is that they cannot be ordered in such a way that is compatible with all the arithmetic operations.

Suppose, for the sake of argument, that there was such an order. Then we either have $i > 0$ or $i < 0$. Let's first consider the case when $i > 0$. If the order is compatible with the other operations, then multiplication by a positive number preserves the order. So $i^2 > 0$. But $i^2 = -1$. So $-1 > 0$. This seems like a contradiction already, but maybe we just have some wierd order that happens to have this property. So, let's just go with it. If $-1 > 0$. Then multiplication by -1 preserves the order. This gives $(-1)(-1) > 0$. So, $1 > 0$. Now we know that $-1 > 0$ and $1 > 0$. But, when you negate both sides of an equation you reverse the order. So $1 > 0$ implies that $-(1) < -(0)$. That is, $-1 < 0$. Now we know that $-1 > 0$ and $-1 < 0$. This is a contradiction. We get to the same kind of contradiction if we assume that $i < 0$. So, since both cases lead to a contradiction, there is no possible order on $\boldsymbol{C}$.

If you think about it, it makes sense that there isn't an ordering on $\boldsymbol{C}$. The reals form a *number line* and there is an obvious geometrical notion of left and right. The complex numbers really form a *number plane* and so there isn't an obvious notion of left and right.

The situation really isn't that bad though. We can think of $\boldsymbol{R}$ as a subset of $\boldsymbol{C}$, just by considering the number

$$a + 0i$$

to be the real number a. When we do this, all the operations above give the usual operations on $\boldsymbol{R}$. Also, the ordering on $\boldsymbol{R}$ still is compatible with all the operations. For example

$$(a_1 + 0i)(a_2 + 0i) = (a_1a_2 + 0) + i(0 + 0) = a_1a_2 + 0i.$$

This shows that multiplication does the right thing on our copy of the reals.

So, the right way to think about it is that $\boldsymbol{C}$ extends $\boldsymbol{R}$ (like a plane extends a line) but it is not possible to extend the ordering on $\boldsymbol{R}$ to an ordering on $\boldsymbol{C}$.

5 Quadratic Polynomials

For starters, it makes sense to take the square root of any real number in $\boldsymbol{C}$. If $D > 0$ then $\sqrt{D}$ is just defined in the usual way. If $D < 0$, then

$$\sqrt{D} = i\sqrt{|D|}.$$

So, for instance $\sqrt{-3} = \pm i\sqrt{3}$. Once we make this definition, the quadratic formula

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

gives roots to the polynomial $Ax^2 + Bx + C$. There are 3 possibilities:

1. The roots are real and unequal. This happens when $B^2 - 4AC > 0$.
2. There is one real root. This happens when $B^2 - 4AC = 0$.
3. The roots are non-real and conjugate. That is, the two roots have the form z and $\overline{z}$. This happens when $B^2 - 4AC < 0$.

Case 2 seems different from the others because there is just one root. But, here's a way to think about it. Imagine varying the coefficients A, B, C of the polynomial so that you pass from Case 1 to Case 3. Then you have these two real roots which approach each other along the real axis, then collide, then split apart again. So, in Case 2, you should think that there are two roots, but that they coincide. They sort of regain their independent identities when they split apart.

The Fundamental Theorem of Algebra says that any polynomial of degree n with complex coefficients, say

$$a_0 + a_1 z + ... + a_n z^n,$$

has n roots, provided that they are counted correctly. Put another way, a typical choice for the coefficients $a_0, ..., a_n$ leads to a polynomial which has n distinct roots, and then the cases where there are fewer roots comes from the same kind of collision problems discussed for quadratic polynomials. I'll give a proof of the Fundamental Theorem of Algebra in the next handout.

6 Geometry of Addition

If we consider complex numbers as points in the plane, then they add like vectors – i.e. coordinatewise. This means that they satisfy the parallelogram rule: Considered as points in the plane, the points 0, z_1, z_2, and $z_1 + z_2$ form the vertices of a parallelogram.

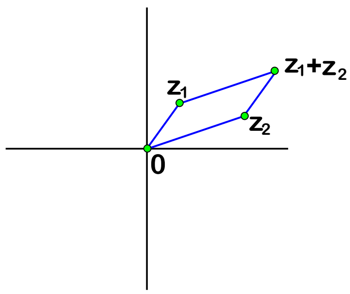

Figure 1: The parallelogram law.

7 Angle Addition Formulas

Let's take a break and recall two formulas from trigonometry. These are called the angle addition formulas.

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2).$$

$$\sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2).$$

I'll give a proof of these formulas below, but first I want to use them.

8 Geometry of Multiplication

A complex number $z = a + bi$ has two important geometric features. First, it has a norm, $|z| = \sqrt{a^2 + b^2}$. In the Figure 2. $|z|$ is denoted by r (for radius.) Second, it has an *argument*, which is the angle that the point (a, b) makes with the x axis. In the picture, the argument of z is the angle θ. This is sometimes written $\arg(z) = \theta$.

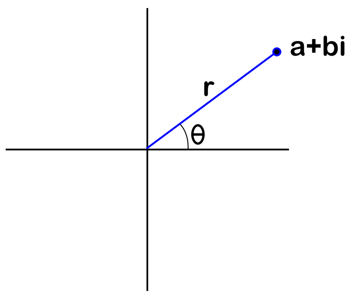

Figure 2: The norm and argument of a complex number

Basically, the quantities r and θ are the polar coordinates of (a, b). Here are the basic facts about complex numbers.

1. $|z_1 z_2| = |z_1||z_2|$.

2. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \bmod 2\pi$.

In other words, when you multiply two complex numbers, their norms multiply and their arguments add. The mod 2π in the formula indicates that angles, as usual, are only defined up to multiples of 2π. For instance π and 3π are considered to be the same angle.

The first formula is easy to derive

$$|z_1 z_2| = \sqrt{z_1 z_2 \overline{z_1 z_2}} = \sqrt{(z_1 \overline{z}_1)(z_2 \overline{z}_2)} = \sqrt{z_1 \overline{z}_1}\sqrt{z_2 \overline{z}_2} = |z_1||z_2|.$$

The second formula uses the angle addition formulas from trig. Actually, the second derivation gets both formulas at the same time.

Suppose that $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are two complex numbers with norms r_1 and r_2 and arguments θ_1 and θ_2. It follows from basic trigonometry that

$$a_1 = r_1 \cos(\theta_1), \qquad b_1 = +r_1(\sin(\theta_1).$$

Therefore,

$$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)).$$

The same goes for z_2. We compute

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) =$$

$$r_1 r_2 \times \Big(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\Big(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\Big)\Big) =$$

$$r_1 r_2 \times \Big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\Big).$$

The last equality comes from the angle addition formulas. When you interpret this last formula geometrically, you see that $z_1 z_2$ has norm $r_1 r_2$ and argument $\theta_1 + \theta_2 \bmod 2\pi$.

You can use the geometric picture to see that lots of other polynomials have roots in $\boldsymbol{C}$. For instance, consider the polynomial $x^5 - 1$. We already know that -1 is a root. Another root is

$$z = \cos(2\pi/5) + i\sin(2\pi/5).$$

This number has norm 1 and argument $2\pi/5$. From the two main properties above, z^5 has norm 1 and argument 2π, which is the same as 0, mod 2π. So $z^5 = 1$. Can you find the other 3 roots?

9 Angle Addition Formulas Revisited

The angle addition formulas are really just equivalent to the statement that when you multiply complex numbers their arguments add. So, if we could somehow prove that complex numbers had this property without using the angle addition formulas, then we would have given a proof of the angle addition formulas.

So, now I'm going to sketch the "arguments add" principle without using the trig formulas. Let's say that a complex number z_1 is *good* if the angle addition formula holds for every pair of numbers (z_1, z_2). Here are some basic facts.

- If z_1 is a positive real number, then $z_1 z_2$ points in exactly the same direction as z_2 because we're just scaling up the coefficients of z_2 by z_1. So, the positive real numbers are all good.

- If z_1 is a negative real number, then $z_1 z_2$ points in exactly the opposite direction as z_2 because, again, we're just scaling up the coefficients of z_2 by z_1. In this case $\arg(z_1) = \pi$, and $\arg(z_1) + \arg(z_2) = \arg(z_2) + \pi$. So, negative real numbers are good.

- If z_1 is good, then rz_1 is also good for for any positive real number r, because $(rz_1)z_2$ and z_1z_2 point in the same direction.
- The number $z_1 = i$ is good, because $z_1z_2 = (-b_2 + ia_2)$. The point $(-b_2, a_2)$ is what you get when you rotate (a_2, b_2) by $\pi/2$ counterclockwise, and $\arg(i) = \pi/2$. So i is good.
- For similar reasons, $-i$ is also good. Applying the previous observations about scaling, the numbers ri and $-ri$ are good for any positive real number r. That is, all the pure imaginary numbers are good. (A pure imaginary number is one of the form $0 + bi$.)

Now we know that the real numbers are all good, and the imaginary numbers are all good. Now for the final step:

Lemma 9.1 *If z_1 and z_1' are both good, then so is $z_1 + z_1'$.*

Proof: The numbers

$$0, \qquad z_1, \qquad z_1'; \qquad z_1 + z_1'$$

satisfy the parallelogram law. They make a parallelogram P. Likewise, the numbers

$$0, \qquad z_1z_2, \qquad z_1'z_2; \qquad z_1z_2 + z_1'z_2$$

satisfy the parallelogram law. They make a parallelogram Q.

We have the distributive law

$$(z_1 + z_1')z_2 = z_1z_1' + z_1'z_2$$

This means that $(z_1 + z_1')z_2$ is also a vertex of Q. But the two sides of Q emanating from 0 are obtained from the two sides of P emanating from 0 by the correct rotation and scaling, since z_1 and z_1' are both good. But these sides determine Q. So, $(z_1 + z_1')z_2$ is obtained from z_2 by the correct rotation and scaling. In other words $z_1 + z_1'$ is also good. ♠

Any complex number is the sum of a real number and an imaginary number. Since real numbers are good and imaginary numbers are good, so is their sum. That is, all complex numbers are good. This proves the angle addition property without (explicitly) using trig.

10 Exercises

Pick 4 out of these and do them.

1. Compute the following quantities

$$(2-i)(3+2i), \qquad \frac{3+4i}{2-3i}, \qquad (1+i)^4.$$

2. Find the roots of the polynomial $z^3 - i$. In other words, find all the complex numbers z such that $z^3 = i$.

3. Say that a complex number z is a unit if $|z| = 1$. Prove that if z and w are units, then so is z/w.

4. Say that a *special unit* is a unit of the form $a + bi$ where a and b are integers. How many special units are there, and why?

5. Say that a *Gaussian integer* is a complex number of the form $a+bi$ where a and b are integers. The prime number 5 factors into "smaller" Gaussian integers:

$$5 = (3+4i)(3-4i).$$

Of all the primes less than 20 which ones factor into smaller Gaussian integers and which ones don't? Do you see a pattern?

6. Consider the function $f(z) = 1/z$. Let L be the line of the form $1 + bi$. This is the vertical line through the point $(1, 0)$ on the x-axis. Define

$$f(L) = \{f(z) | z \in L\}.$$

In other words, you want to apply the function f to all the complex numbers in L and see what you get. What shape is $f(L)$?

7. Let N be any positive integers. Prove that the polynomial $z^N - 1$ has exactly N roots in $\boldsymbol{C}$.