Leibniz's Formula: Below I'll derive the series expansion

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}; \qquad 0 \le x \le 1. \tag{1}$$

Plugging the equation $\pi = 4\arctan(1)$ into Equation 1 gives Leibniz's famous formula for π, namely

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} \cdots \tag{2}$$

This series has a special beauty, but it is terrible for actually computing the digits of π. For instance, you have to add up about 500 terms just to compute that $\pi = 3.14....$

Machin's Formula: Machin's formula also uses Equation 1, but takes advantage that the series converges much faster when x is closer to 0. Below I'll derive the identity

$$\pi = 16\arctan(1/5) - 4\arctan(1/239). \tag{3}$$

Combining Equations 1 and 3, we get Machin's formula:

$$\pi = \sum_{n=0}^{\infty} (-1)^n A_n, \qquad A_n = \frac{16\ (1/5)^{2n+1} - 4\ (1/239)^{2n+1}}{2n+1}. \tag{4}$$

How fast is Machin's formula? Let S_n be the sum of the first n terms of this series. The series is alternating and decreasing, so that

$$A_n - A_{n+1} = |S_{n+2} - S_n| < |\pi - S_n| < |S_{n+1} - S_n| = A_n \tag{5}$$

Some fooling around with the terms in Equation 4 leads to the bounds

$$A_n < \frac{2}{n25^n}, \qquad A_n - A_{n+1} > \frac{1}{n25^n}.$$

Therefore

$$\frac{1}{n25^n} < |\pi - S_n| < \frac{2}{n25^n} \tag{6}$$

Equation 6 gives a good idea of how fast Machin's method is. For instance, if you add up the first 100 terms in Equation 4, you get about 140 digits of π.

Proof of Equation 3: Call a complex number $z = x + iy$ *good* if $x > 0$ and $y > 0$. For a good complex number z, let $A(z) \in (0, \pi/2)$ be the angle that the ray from 0 to z makes with the positive x-axis. By definition of the arc-tangent,

$$A(x + iy) = \arctan(y/x). \tag{7}$$

If z_1 and z_2 and $z_1 z_2$ are all good, then

$$A(z_1 z_2) = A(z_1) + A(z_2). \tag{8}$$

This is a careful statement of the principle that "angles add when you multiply complex numbers".

A direct calculation establishes the following strange identity:

$$(5 + i)^4 = (2 + 2i)(239 + i). \tag{9}$$

Combining this with several applications of Equation 7 and 8, you get

$$4\arctan(1/5) = \arctan(1) + \arctan(1/239). \tag{10}$$

Rearranging Equation 10, multiplying by 4, and using $4\arctan(1) = \pi$, we get Equation 3.

Proof of Equation 1: When $|y| < 1$ we have the geometric series

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 ... \tag{11}$$

Now substitute in $y = -t^2$, to get

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 ... = \sum_{n=0}^{\infty} (-1)^n t^{2n}, \qquad |t| < 1. \tag{12}$$

Here is the one part of the proof that is really surprising. It is one of the miracles of calculus.

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt, \qquad x \in [0, 1]. \tag{13}$$

I'll derive this equation below.

Combining everything, we get the result:

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \tag{14}$$

The Arctan Function:
Define the functions

$$A(x) = \arctan(x), \quad S(x) = \sin(x), \quad C(x) = \cos(x), \quad T(x) = \tan(x). \tag{15}$$

We have

$$T \circ A(x) = x, \qquad C \circ A(x) = \frac{1}{\sqrt{1+x^2}}, \qquad S \circ A(x) = \frac{x}{\sqrt{1+x^2}}. \tag{16}$$

The first of these is the definition of the arctan (or inverse tangent) function. The second two are forced by the first one, and by the fact that $T = S/C$ and $C^2 + S^2 = 1$.

Applying the Chain Rule to the first equation in Equation 16, we get

$$T'(A(x))A'(x) = (T \circ A)'(x) = 1 \tag{17}$$

Therefore

$$A'(x) = \frac{1}{T'(A(x))}. \tag{18}$$

By the quotient rule,

$$T' = \left(\frac{S}{C}\right)' = \frac{S'C - C'S}{C^2} = \frac{C^2 + S^2}{C^2} = \frac{1}{C^2}. \tag{19}$$

Combining the last three equations, we get

$$A'(x) = \big(C \circ A(x)\big)^2 = \frac{1}{1+x^2}. \tag{20}$$

Since $A(0) = 0$, Equation 13 follows from the last equation and the Fundamental Theorem of Calculus.