Final Exam Solutions

1

$$\int_{-1}^{1} \frac{dx}{1+x^2} = \arctan(1) - \arctan(-1) = \pi/4 - (-\pi/4) = \pi/2.$$

$$\int_{-1}^{1} \frac{1+\cos(x)\sin^3(x)}{1+x^2} dx =$$

$$\int_{-1}^{1} \frac{dx}{1+x^2} + \int_{-1}^{1} g(x)dx = \pi/2 + \int_{-1}^{1} g(x)dx = \pi/2 + 0 = \pi/2.$$

The point is that g is an odd function. So, the integral of g from -1 to 1 is zero.

2. The region lies between the graphs of the functions $f(x) = x$ and $g(x) = \sqrt{1-x^2}$. The boundaries are $x = 0$ and $x = \sqrt{2}/2$. So, using washers, we get that the volume is

$$\pi \int_{0}^{\sqrt{2}/2} \left((1-x^2) - x^2\right)\, dx = \pi\sqrt{2}/3.$$

3. This is a linear equation with $P(x) = -2$ and $Q(x) = 5\sin(x)$. Compute

$$I(x) = e^{-2x}.$$

This gives the solution as

$$y(x) = \frac{\int I(x)Q(x)dx}{I(x)} = \frac{5\int \sin(x)e^{-2x}\, dx}{e^{-2x}} = -\cos(x) - 2\sin(x) + Ce^{2x}.$$

Plugging in $y(0) = 0$ gives $C = 1$. So, the solution is

$$y(x) = \cos(x) - 2\sin(x) + e^{2x}.$$

That means $y(\pi/2) = -2 + e^{\pi}$.

4. This is a separable equation. Separate variables to get

$$\int \frac{dy}{1+y^2} = \int \frac{\sin(x)dx}{1+\cos^2(x)}.$$

The integral on the right is done using the u-sub $u = \cos(x)$. This leads to

$$\arctan(y) = -\arctan(\cos(x)) + C.$$

The initial conditions give $C = 0$. So, the solution

$$y = \tan(-\arctan(\cos(x)) = -\cos(x).$$

So, $y(2) = \cos(2)$.

5. Note that $n + 7 < 2n$ for n large and $3 - 1/n < 2.5$ for n large. So,

$$\frac{n+7}{3^{3-1/n}+5} < \frac{2n}{n^{2.5}} = \frac{2}{n^{1/5}}.$$

The series

$$\sum \frac{2}{n^{1.5}}$$

converges by the p-test (for $p = 1.5$.) So, by the comparison test, the original series converges.

6. 6 a. Integration by parts (using $u = x$ and $dx = e^{-x}$) gives

$$\int_0^\infty xe^{-x}\, dx = 1.$$

b. The integrand in the second integral is positive for large x and less than $2xe^{-x}$. So, by the integral comparison test and part a, the integral converges.
c. The nth term in the series here is less than $2ne^{-n}$. The series

$$\sum 2ne^{-n}$$

converges by the integral test and part a. The original series therefore converges by the comparison test.

7. The ratio test easily gives the radius of convergence as $1/e$. At the endpoint $x = 2 - 1/e$, the series is alternating and decreasing, and the terms limit to 0. So, the series converges for $x = 2 - 1/e$. For $x = 2 + 1/e$ you have the harmonic series, which diverges. So, the interval of convergence is $[2 - 1/e, 2 + 1/e)$.

8. Here are the steps for the first series:

$$\frac{1}{1+t} = 1 + t + t^2 + t^3 ...$$

$$\frac{t}{1+t} = t + t^2 + t^3 + ...$$

Substituting in t^3 for t gives

$$\frac{t^3}{1+t^3} = t^3 + t^6 + t^9 + ...$$

Integrating term by term gives

$$\int \frac{x^3}{1+x^3} = \frac{x^4}{4} + \frac{x^7}{7} + \frac{x^{10}}{10} + ... = \sum_{n=0}^{\infty} \frac{x^{3n+4}}{3n+4}.$$

Here are the steps for the second series. First imitate the beginning of partial fractions, to write

$$\frac{5x-4}{2x^2+x-1} = \frac{3}{1+x} - \frac{1}{-1+2x} = \frac{3}{1+x} + \frac{1}{1-2x}.$$

Using similar steps to part a, we have

$$\frac{3}{1+x} = 3 - 3x + 3x^2 - 3x^3 + 3x^4 ... = \sum_{n=0}^{\infty} (-1)^n 3) x^n.$$

and

$$\frac{1}{1-2x} = 1 + 2 + 4x^2 + 8x^3 + 16x^4 ... = \sum_{n=0}^{\infty} 2^n x^n.$$

So, adding these two series, you get

$$\sum_{n=0}^{\infty} (2^n + (-1)^n 3) x^n$$

for a final answer.

9. Let $f(x) = \sin^2(x)$. You can compute

$$f'(x) = 2\cos(x)\sin(x)$$

$$f''(x) = 2(\cos^2(x) - \sin^2(x))$$
$$f'''(x) = -8\cos(x)\sin(x)$$
$$f^{(4)}(x) = -8(\cos^2(x) - \sin^2(x)),$$

The pattern repeats in the obvious way. The odd terms vanish and we have

$$f^{2n}(0) = (-1)^n 2^{2n-1}.$$

Plugging this into the basic formula, you get the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n-1}}{(2n)!} x^{2n}.$$

You can also do this problem by using the half-angle formula and the series for cos.

10a. Let

$$g(t) = e^{\sin(t)}$$

Then we're interested in the first 4 terms of $f(t) = tg(t)$. This means we just need the first 3 terms of $g(t)$. We compute

$$g(0) = 1.$$

$$g'(t) = e^{\sin(t)}\cos(t), \qquad g'(0) = 1.$$
$$g''(t) = e^{\sin(t)}\cos^2(t) - e^{\sin(t)}\sin(t), \qquad g''(0) = 1.$$

So, the series for $g(t)$ starts out

$$1 + t + t^2/2.$$

This means that the series for $f(t)$ starts out

$$t + t^2 + t^3/2.$$

b Let

$$F(x) = \int_0^x f(t)dt.$$

We want to estimate $F(1)$. Integrating the beginning of the series for $f(t)$ term by term gives

$$F(x) \approx x^2/2 + x^3/3 + x^4/8.$$

Taylor's Remainder Theorem gives

$$|F(1) - (1/2 + 1/3 + 1/8)| \leq \frac{M \times 1^5}{5!}.$$

Here M is the maximum value taken by $|F^{(5)}|$ on $[0, 1]$. So, $F(1) = 23/24$ plus some error which is less than $M/120$. To estimate M you would need to take the first 5 derivatives of F, which is the same as the first 4 derivatives of f, and estimate the maximum on the interval $[0, 1]$.