Machin's Method of Computing the Digits of π

Here is the identity that goes into Machin's formula for estimating π.

$$\pi = 4\arctan(1) = 16\arctan(1/5) - 4\arctan(1/239). \tag{1}$$

Below, I'll explain how this is derived.

We also have the series expansion

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - x^3/3 + x^5/5 - x^7/7... \tag{2}$$

The radius of convergence of this series is 1. Below, I'll derive this series.

These two equations are all you need to know for Machin's method of estimating π. Combining Equation 1 and 2, we get

$$\pi = 16 \sum_{n=0}^{\infty} (-1)^n \frac{(1/5)^{2n+1}}{2n+1} - 4 \sum_{n=0}^{\infty} (-1)^n \frac{(1/239)^{2n+1}}{2n+1}. \tag{3}$$

If you want to write it as a single series, then you get

$$\pi = \sum_{n=0}^{\infty} (-1)^n \frac{16\ (1/5)^{2n+1} - 4\ (1/239)^{2n+1}}{2n+1}. \tag{4}$$

Let s_n be the sum of the first n terms of this series. The series is alternating and decreasing, so the nth remainder is less than the $(n+1)$st term. That is,

$$|\pi - s_n| < \frac{16\ (1/5)^{2n+3} - 4\ (1/239)^{2n+3}}{2n+3}. \tag{5}$$

For instance, a little calculation shows that $|\pi - s_{100}| < 10^{-140}$. So, if you sum up the first 100 terms of Equation 4, you get the first 140 digits of π.

The bound on the right hand side of Equation 4 is pretty messy, but notice that it is a little bit less than $(1/25)^n$. So, a simpler bound is

$$|\pi - s_n| < (1/25)^n. \tag{6}$$

Every time you add another term in the series, you get an approximation to π which is about 25 times more accurate!

Proof of Equation 1: Call a complex number $z = x + iy$ *good* if $x > 0$ and $y > 0$. For a good complex number z, let $A(z)$ be the angle that the ray from 0 to z makes with the positive x-axis. The angle is always taken to lie in $(0, \pi/2)$. From basic trigonomatry, you get the formula

$$A(x + iy) = \arctan(y/x). \tag{7}$$

If z_1 and z_2 and $z_1 z_2$ are all good, then

$$A(z_1 z_2) = A(z_1) + A(z_2). \tag{8}$$

This is a careful statement of the principle that "angles add" when you multiply complex numbers.

Rather strangely, it turns out (and you can just compute by hand) that

$$(5 + i)^4 = (2 + 2i)(239 + i). \tag{9}$$

Combining this with several applications of Equation 8, you get

$$4 \arctan(1/5) = \arctan(1) + \arctan(1/239).$$

Rearranging the equation and multiplying by 4, we get Equation 1

Proof of Equation 2: All the equations we write are valid for $|x| < 1$. We have the geometric series

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 ... \tag{10}$$

Now substitute in $-x^2$ for x. This gives

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 ... \tag{11}$$

Integrating both sides of this equation gives Equation 2.