Dealing with Trig Integrals

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1 Step 1: Convert to sines and cosines

In general, we're going to integrate

$$I(m,n) := \int \cos(x)^m \sin(x)^n dx \tag{1}$$

where m and n are integers.

If the integral involves other trig functions, convert to sines and cosines. For instance,

$$\int \csc^5(x)\cot^4(x)dx = \int \frac{1}{\sin^5(x)}\,\frac{\cos^4(x)}{\sin^4(x)} = I(4,-9).$$

2 The Swap

Since $\sin(\pi/2 - x) = \cos(x)$ and $\cos(\pi/2 - x) = \sin(x)$, the substitution $u = \pi/2 - x$ and $du = -dx$ leads to the following formula, which I'll call *the swap*.

$$I(b,a) = -\overline{I}(a,b). \tag{2}$$

The notation $\overline{I}(a,b)$ means that you should take the final expression, and make the swaps $\sin(x) \leftrightarrow \cos(x)$ and $\tan(x) \leftrightarrow \cot(x)$ and $\sec(x) \leftrightarrow \csc(x)$. For instance

$$I(0,-1) = -\overline{I}(-1,0) = -\text{swap}\Big(\int \sec(x)dx\Big) =$$

$$-\text{swap}(\ln|\sec(x)+\tan(x)|) = -\ln|\csc(x)+\cot(x)|.$$

3 Step 2: Positive, Odd

Suppose $m > 0$ is odd. Here's what you do, by way of example.

$$I(7,-11) = \int \cos^6(x)\sin^{-11}(x)\cos(x)dx =$$

$$\int (1-\sin^2(x))^3 \sin^{-11}(x)\cos(x)dx = \int (1-u^2)^3 u^{-11} du.$$

The last step used $u = \sin(x)$ and $du = \cos(x)dx$. Expand out the last integral, integrate, then switch back to x.

If $n > 0$ is odd, use the swap to switch to the case when $m > 0$ is odd.

4 Step 3: Zero

Suppose that $m = 0$, so you just have $I(0,n)$. Integration by parts gives

$$n\ I(0,n) = (n-1)\ I(0,n-2) - \cos(x)\sin^{n-1}(x) + C. \tag{3}$$

Using this formula, you can reduce to the cases $n = -1, 0, 1$. The cases $n = 0, 1$ are easy, and

$$I(0,-1) = -\ln|\csc(x) + \cot(x)| + C.$$

When n is odd and positive, you can use Step 2 as a shortcut.

If $n = 0$ you can use the swap to get back to the case $m = 0$.

5 Step 4: Positive, Even

Suppose that $m > 0$ is even. Here is what you do, by way of example.

$$I(6,-5) = \int (1-\sin^2(x))^3 \sin^{-5} dx =$$

$$\int \sin^{-5}(1 - 3\sin^2(x) + 3\sin^4(x) - \sin^6(x))dx =$$

$$I(0,-5) - 3I(0,-3) + 3I(0,-1) - I(0,1).$$

Each individual integral is then done by Step 3. If $n > 0$ is even, use the swap to get back to the case that $m > 0$ is even.

6 Step 5: Negative Even Sum

So far, we've handled the cases where either $m \geq 0$ or $n \geq 0$. We just have to deal with the cases where both are negative. If $m + n$ is even, there is a trick, which I'll illustrate by way of example.

$$I(-7, -3) = \int \sec^{10}(x) \tan^{-3}(x) dx = \int \sec^{8}(x) \tan^{-3}(x) \sec^{2}(x) dx =$$

$$\int (1 + \tan^2(x))^4 \tan^{-3}(x) \sec^2(x) dx = \int (1 + u^2)^4 u^{-3} du.$$

The last step used $u = \tan(x)$ and $du = \sec^2(x)dx$. Expand out the last integral, integrate, then switch back to x.

7 Step 6: Negative Odd Sum

The last case is where m and n are negative and (exactly) one of them is even. Use the swap to get the case where m is even. Integration by parts gives the reduction law

$$(m + 1)\ I(m, n + 2) = (n + 1)\ I(m + 2, n) - \cos^{m+1}(x) \sin^{n+1}(x) + C. \quad (4)$$

Use the formula repeatedly to reduce to the case $m = 0$, then apply Step 3.