M1410 Homework Assignment 1:

The purpose of this assignment is to guide you through a proof that the mixed second partials of a function $F : \boldsymbol{R}^n \to \boldsymbol{R}$ are equal when they are continuous. All the problems use the following terminology. Mostly we'll work with $n = 2$. One disadvantage to this outline is that the first problem is possibly the hardest.

Defintion: Say that $F : \boldsymbol{R}^2 \to \boldsymbol{R}$ is *special* if F has continuous second partial derivatives, and F vanishes on the coordinate axes. That is, $F(t, 0) = F(0, t) = 0$ for all t.

1. Suppose F is special and

$$\frac{\partial^2 F}{\partial x \partial y}(0, 0) = 0.$$

Prove that

$$\lim_{t \to 0} \frac{F(t, t)}{t^2} = 0. \tag{1}$$

Hint: First show that

$$|F(t, t)| \leq t \times \sup_{s \in [0,t]} |\partial f / \partial y(t, s)|.$$

2. Suppose that F is special and

$$\frac{\partial^2 F}{\partial x \partial y}(0, 0) = C.$$

$$\lim_{t \to 0} \frac{F(t, t)}{t^2} = C. \tag{2}$$

(Hint: Apply Exercise 1 to the function $G(x, y) = G(x, y) - Cxy$, which is again special.

3: Prove that the second mixed partials of a special function are equal at the origin. Hint: use Exercise 2.

4. Let $\mathcal{V}$ denote the set of functions on $\boldsymbol{R}^2$ whose second mixed partials exist and are equal at the origin. Prove that V is a real vector space. (The addition law is just $(f+g)(x) = f(x)+g(x)$). It follows from Exercise 3 that V contains all special functions.

5. Say that a function $G : \boldsymbol{R}^2 \to \boldsymbol{R}$ is *simple* if one of the following two properties holds:

- $G(x, y) = F(x, 0)$ for some function F having first partial derivatives.
- $G(x, y) = F(0, y)$ for some function F having first partial derivatives.

Prove that the second mixed partials of a simple function are 0. Hence, the vector space $\mathcal{V}$ contains all simple functions, and all finite sums of simple functions.

6. Let $F : \boldsymbol{R}^2 \to \boldsymbol{R}$ be a function whose second partials exist and are continuous. Prove that F is the sum of a special function and finitely many (in fact three) simple functions. Hence $F \in \mathcal{V}$.

7. Let $F : \boldsymbol{R}^n \to \boldsymbol{R}$ be a function whose second partials exist and are continuous. Prove that

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i},$$

for all i and j. Hint: reduce this to the case $n = 2$, and then compose F with suitable translations. In other words, if you can prove something at the origin for all functions, you can prove the same thing for all functions at all points.