M1410 Homework Worksheet:

1. Let $M \in SO(n)$ be an orthogonal $n \times n$ matrix. Let M_{ij} be an entry of M, and let μ_{ij} be the $(n-1) \times (n-1)$ minor of M obtained by crossing off the row and column through M_{ij}. Prove that $M_{ij} = (-1)^{i+j} \det(\mu_{ij})$.

2. Use Problem 1 to prove the following result without applying the rotational symmetry of the star operator: Suppose $w_1, ..., w_n$ is an orthonormal basis of $\boldsymbol{R}^n$. Then $*(dw_1) = \pm dw_2 \wedge ... \wedge dw_n$.

3. Do the following variant of Problem 8 on P 319 of the book: Suppose that M is a matrix with all positive entries. Then there exists an eigenvector for M which has all positive entries. Hint: Consider the action of M on the space of lines through the origin in $\boldsymbol{R}^n$ which point into the positive orthant and use the Brouwer fixed point theorem.

4. Do Problem 11 on P 319.

5. Suppose that Q is a Riemannian metric on $\boldsymbol{R}^2$ with the property that there is some smooth positive function ρ on $\boldsymbol{R}^2$ such that

$$Q_p(v, w) = \rho(p)\ v \cdot w$$

for all vectors v, w and all points p. In other words, Q_p is just a multiple of the dot product at each point, through the multiple might change with the point. We can define the Hodge star operator with respect to Q just as as for the Euclidean metric. A function f is harmonic with respect to Q if $d * d * f = 0$ at all points. Here $*$ is the version of the star operator defined relative to Q. Prove that, for 1-forms, the definition of $*$ is the same for Q as it is for the Euclidean metric. For this reason, a function is harmonic with respect to Q if and only if it is harmonic in the ordinary sense.