M1410 Homework Worksheet:

1. Consider the real valued function

$$f(x) = e^{-1/(x^2)}.$$

This function is defined whenever $x > 0$. Extend f so that $f(x) = 0$ whenever $x \leq 0$. Prove that f is a smooth function, in spite of its piecemeal definition.

2. Using the function from problem 1, show that there is a smooth function $f : \boldsymbol{R} \to \boldsymbol{R}$ such that $f(x) \geq 1$ when $|x| \leq 1$ and $f(x) = 0$ when $|x| \geq 2$. Such a function is called a bump function, though techically one often requires $f(x) = 1$ when $|x| \leq 1$. This stronger condition, not needed for the remaining problems, is a bit harder to arrange.

3. Using the function from problem 2, prove the following result. Let Δ_1 and Δ_2 be any two balls in $\boldsymbol{R}^n$ with the closure of Δ_1 contained in the interior of Δ_2. Then there is a function $f : \boldsymbol{R}^n \to \boldsymbol{R}$ such that $f(p) \geq 1$ for $p \in \Delta_1$ and $f(p) = 0$ for $p \in \boldsymbol{R}^n - \Delta_2$.

4. Let M be a compact k-dimensional manifold. Prove that there is a finite collection of coordinate patches $U_1, ..., U_n$ and a finite collection of smooth functions $f_1, ..., f_n$ on M such that

- $f_j(p) = 0$ when $p \notin U_j$.
- $f_1(p) + ... + f_n(p) = 1$ for all $p \in M$.

The collection $(U_1, ..., U_n; f_1, ..., f_n)$ is called a *partition of unity*. A coordinate patch is the image of an open set in $\boldsymbol{R}^k$ under a coordinate chart. Hint: use problem 3.

Bonus Problem 1: Use the bump function technology to prove the following result. Suppose that $p_1, ..., p_n$ is a finite collection of points in $\boldsymbol{R}^2$ and π is some permutation of these points. Then there is a diffeomorphism of $\boldsymbol{R}^2$ which permutes the points as π does.

bonus Problem 2: Use the bump function technology to show that that every compact smooth manifold has a smooth Riemannian metric.